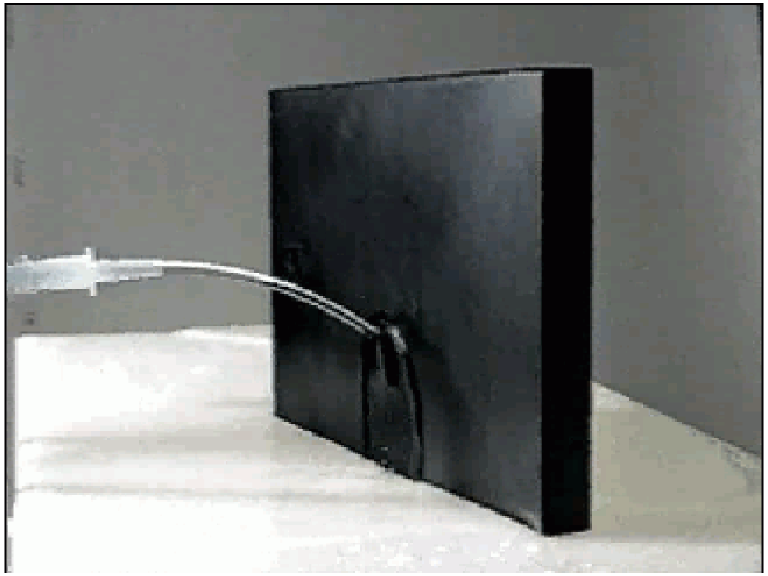
Lecture 4

Differential Analysis of Fluid Flow

Navier-Stockes equation

Newton second law and conservation of momentum & momentum-of-momentum

A jet of fluid deflected by an object puts a force on the object. This force is the result of the change of momentum of the fluid and can happen even though the speed (magnitude of velocity) remains constant.



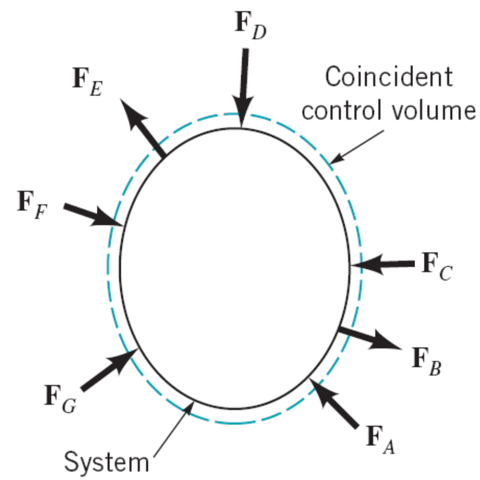
Newton second law and conservation of momentum & momentum-of-momentum

In an inertial coordinate system:

$$\boxed{\frac{D}{Dt}\int_{sys} \mathbf{V}\rho d\forall = \sum F_{sys}}$$

Rate of change of the momentum of the system

Sum of all external forces acting on the system



At a moment when system coincide with control volume:

$$\sum F_{sys} = \sum F_{\substack{contents\ of \\ the\ control\ volume}}$$

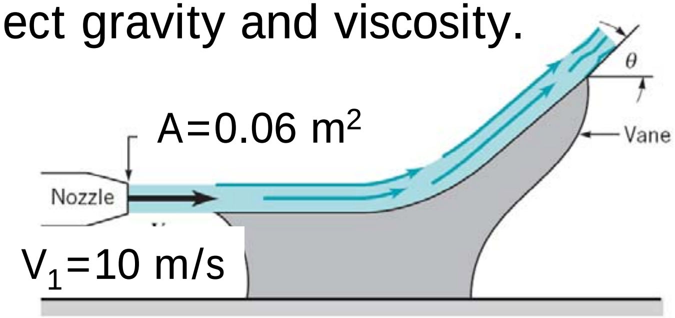
On the other hand: $$\frac{D}{Dt}\int_{sys} \mathbf{V}\rho d\forall = \frac{\partial}{\partial t}\int_{CV} \mathbf{V}\rho d\forall + \int_{CS} \mathbf{V}\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA$$

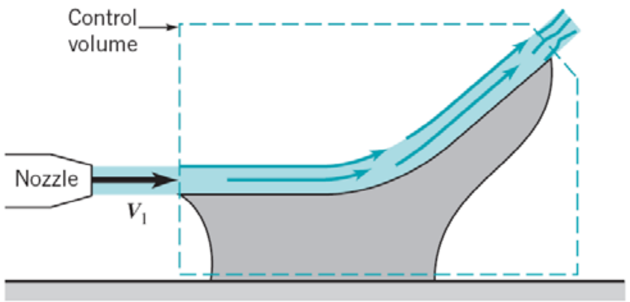
$$\boxed{\frac{\partial}{\partial t}\int_{CV} \mathbf{V}\rho d\forall + \int_{CS} \mathbf{V}\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA = \sum F_{\substack{contents\ of \\ the\ control\ volume}}}$$

Example: Linear momentum

Determine anchoring forces required to keep the vane stationary vs angle Θ.

Neglect gravity and viscosity.





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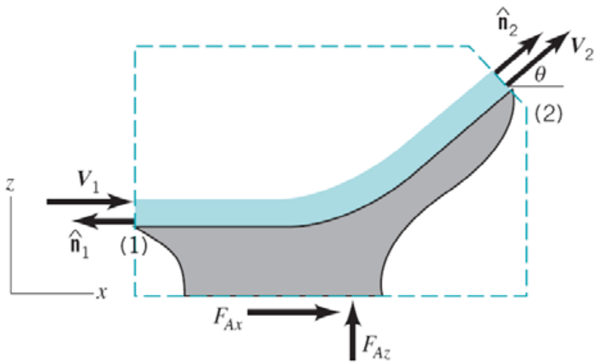

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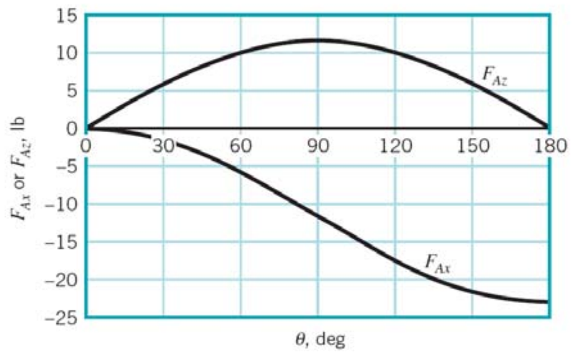


$$\cancel{\frac{\partial}{\partial t}\int_{CV} u\rho\, dV} + \int_{CS} \rho u \mathbf{V}\cdot\mathbf{n}\, dA = \sum F_x$$

$$\cancel{\frac{\partial}{\partial t}\int_{CV} w\rho\, dV} + \int_{CS} \rho w \mathbf{V}\cdot\hat{\mathbf{n}}\, dA = \sum F_y$$

$$\Rightarrow$$

$$V_1\rho(-V_1)A_1 + V_1\cos\theta\rho(V_1)A_2 = F_{Ax}$$

$$0\rho(-V_1)A_1 + V_1\sin\theta\rho(V_1)A_2 = F_{Az}$$

$$\Downarrow$$

$$F_{Ax} = -V_1^2\rho A_1(1-\cos\theta)$$

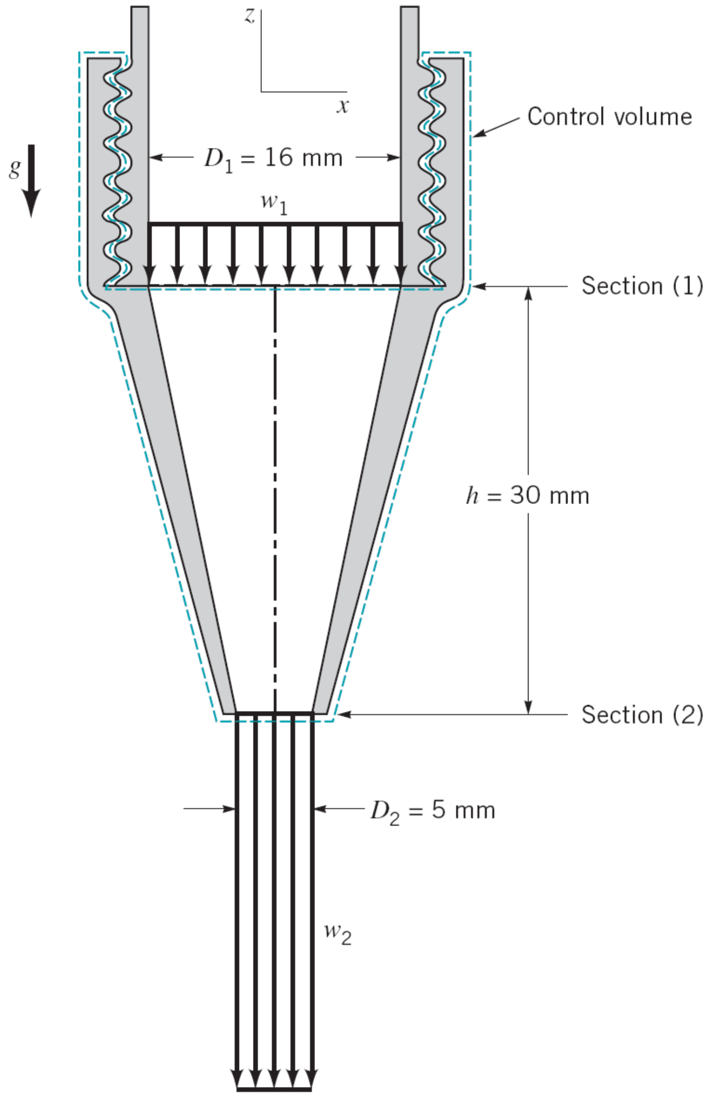
$$F_{Az} = V_1^2\rho A_1\sin\theta$$

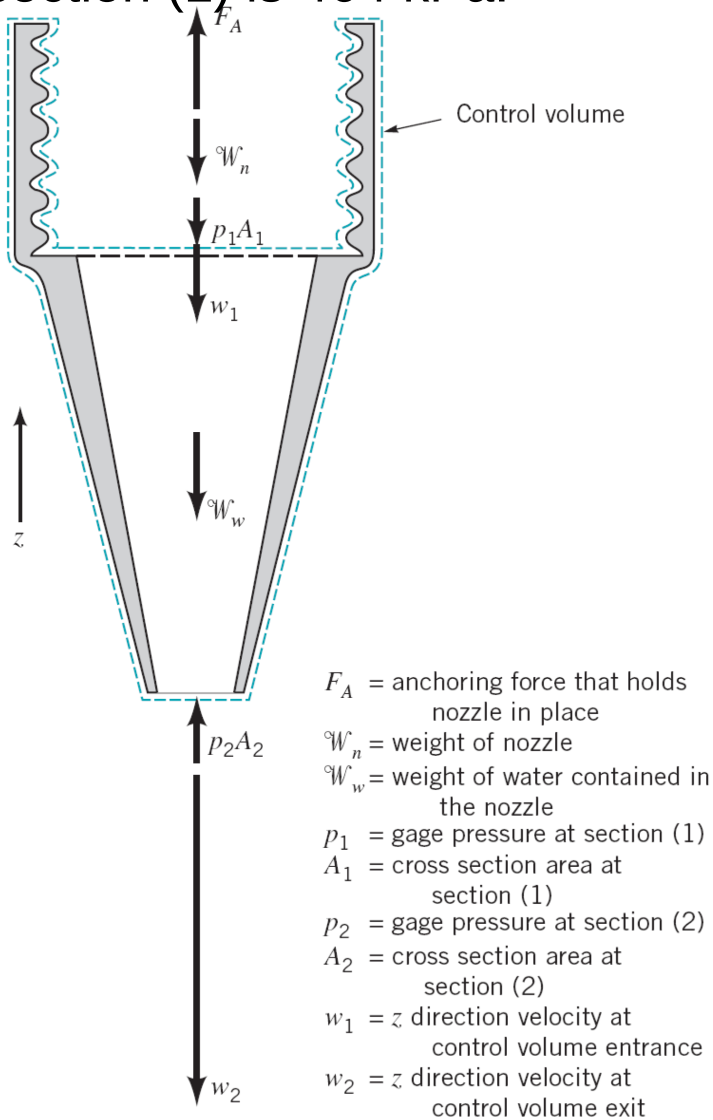
Linear momentum: comments

- Linear momentum is a vector
- As normal vector points outwards, momentum flow inside a CV involves negative $\mathbf{V} \cdot \mathbf{n}$ product and moment flow outside of a CV involves a positive $\mathbf{V} \cdot \mathbf{n}$ product.
- The time rate of change of the linear momentum of the contents of a nondeforming CV is zero for steady flow
- Forces due to atmospheric pressure on the CV may need to be considered

Example: Linear momentum – taking into account weight, pressure and change in speed

- Determine the anchoring force required to hold in place a conical nozzle attached to the end of the laboratorial sink facet. The water flow rate is 0.6 l/s, nozzle mass 0.1kg. The pressure at the section (1) is 464 kPa.

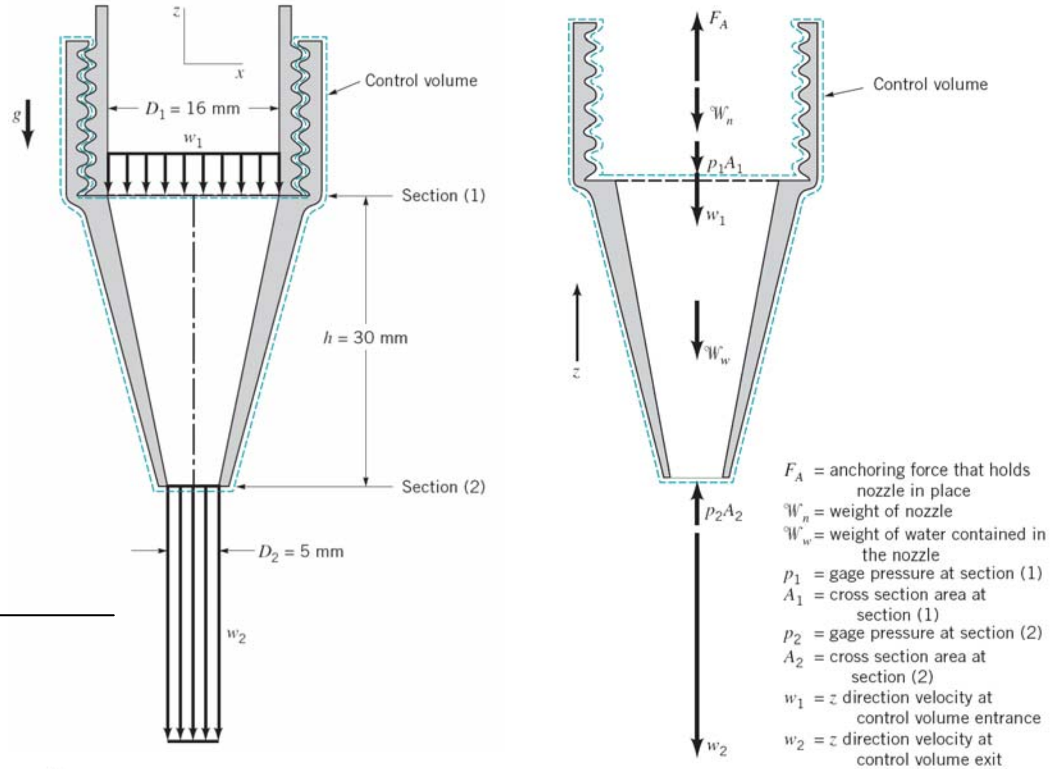




Example: Linear momentum – taking into account weight, pressure and change in speed

$$\frac{\partial}{\partial t}\int_{CV} w\rho\, d\forall + \int_{CS} w\rho\,\mathbf{V}\cdot\hat{\mathbf{n}}\, dA =$$

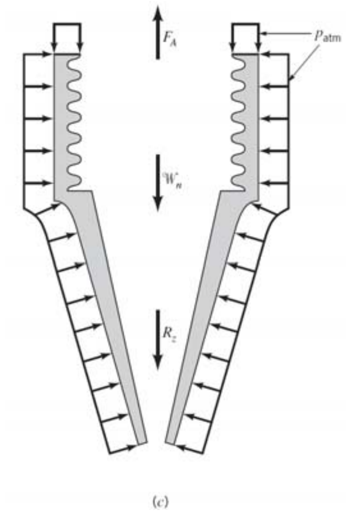
$$= F_A - W_n - p_1 A_1 - W_w + p_2 A_2$$



volume of a truncated cone:

$$\forall_w = \frac{1}{12}\pi h\left(D_1^2 + D_2^2 + D_1 D_2\right)$$

pressure distribution:



Moment-of-Momentum Equation

The net rate of flow of moment-of-momentum through a control surface equals the net torque acting on the contents of the control volume.

Water enters the rotating arm of a lawn sprinkler along the axis of rotation with no angular momentum about the axis. Thus, with negligible frictional torque on the rotating arm, the absolute velocity of the water exiting at the end of the arm must be in the radial direction (i.e., with zero angular momentum also). Since the sprinkler arms are angled "backwards", the arms must therefore rotate so that the circumferential velocity of the exit nozzle (radius times angular velocity) equals the oppositely directed circumferential water velocity.

The Energy Equation

- The First Law of thermodynamics

$$\frac{D}{Dt}\int_{sys} e\rho \, d\forall = (\dot{Q} + \dot{W})_{sys}$$

Rate of increase of the total stored energy of the system

Net rate of energy addition by heat transfer into the system

Net rate of energy addition by work transfer into the system

Total stored energy per unit mass: $e = \hat{u} + \frac{V^2}{2} + gz$

$$\frac{D}{Dt}\int_{sys} e\rho d\forall = \frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS} e\rho \mathbf{V}\cdot\hat{\mathbf{n}}dA$$

Rate of increase of the total stored energy of the system

rate of increase of the total stored energy of the control volume

Net rate of energy flow out of the control volume

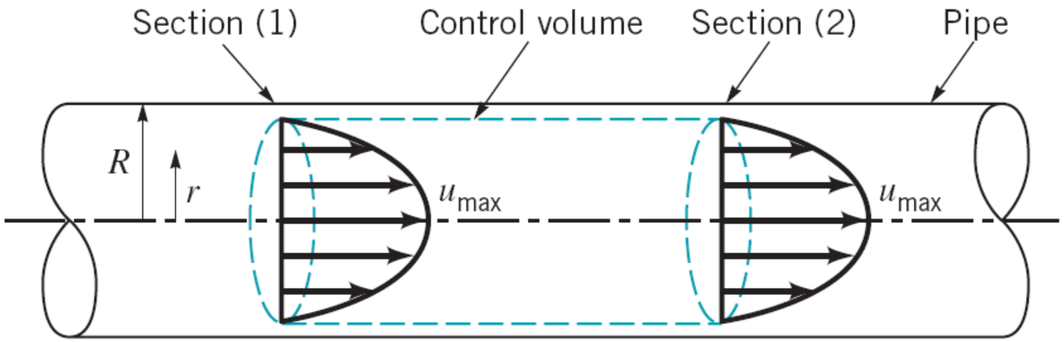
$$\frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS} e\rho \mathbf{V}\cdot\hat{\mathbf{n}}dA = (\dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{net\\in}})_{cv}$$

Power transfer due to normal and tangential stress

- Work transfer rate (i.e. power) can be transferred through a rotating shaft (e.g. turbines, propellers etc)

$$\dot{W}_{shaft} = T_{shaft} w$$ where $\boldsymbol{T_{shaft}}$ – torque and $\boldsymbol{w}$ – angular velocity

- or through the work of normal stress



on a single particle: $$\delta \dot{W}_{\substack{normal\\stress}} = \delta \vec{F}_{\substack{normal\\stress}} \cdot \vec{V} = \sigma \vec{n} \delta A \cdot \vec{V} = -p\vec{V} \cdot \vec{n} \delta A$$

integrating: $$\dot{W}_{\substack{normal\\stress}} = \int_{CS} -p\vec{V} \cdot \vec{n} dA$$

tangential stress: $$\delta \dot{W}_{\substack{\text{tangential}\\\text{stress}}} = \delta \vec{F}_{\substack{\text{tangential}\\\text{stress}}} \cdot \vec{V} = 0$$

Power transfer due to normal and tangential stress

- The first law of themrodynamics can be expressed now as:

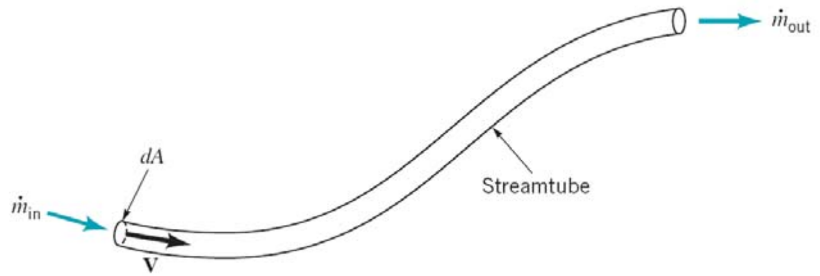
$$\frac{\partial}{\partial t}\int_{CV} e\rho d\mathcal{V} + \int_{CS} e\rho \mathbf{V}\cdot\widehat{\mathbf{n}}dA = \dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{\text{shaft}\\\text{net in}}} - \int_{CS} p\vec{V}\cdot\vec{n}dA$$

- so, we can obtain the **energy equation**

$$\frac{\partial}{\partial t}\int_{CV} e\rho d\mathcal{V} + \int_{CS}\left(\breve{u} + \frac{p}{\rho} + \frac{V^2}{2} + gz\right)\rho\mathbf{V}\cdot\widehat{\mathbf{n}}dA = \dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{\text{shaft}\\\text{net in}}}$$

Application of energy equation

- Let's consider a steady (in the mean, still can be cyclical) flow and take a one stream
- Product **V·n** is non-zero only where liquid crosses the CS; if we have only one stream entering and leaving control volume:



$$\frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS}\left(\breve{u}+\frac{p}{\rho}+\frac{V^2}{2}+gz\right)\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA = \dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{\text{shaft}\\\text{net in}}}$$

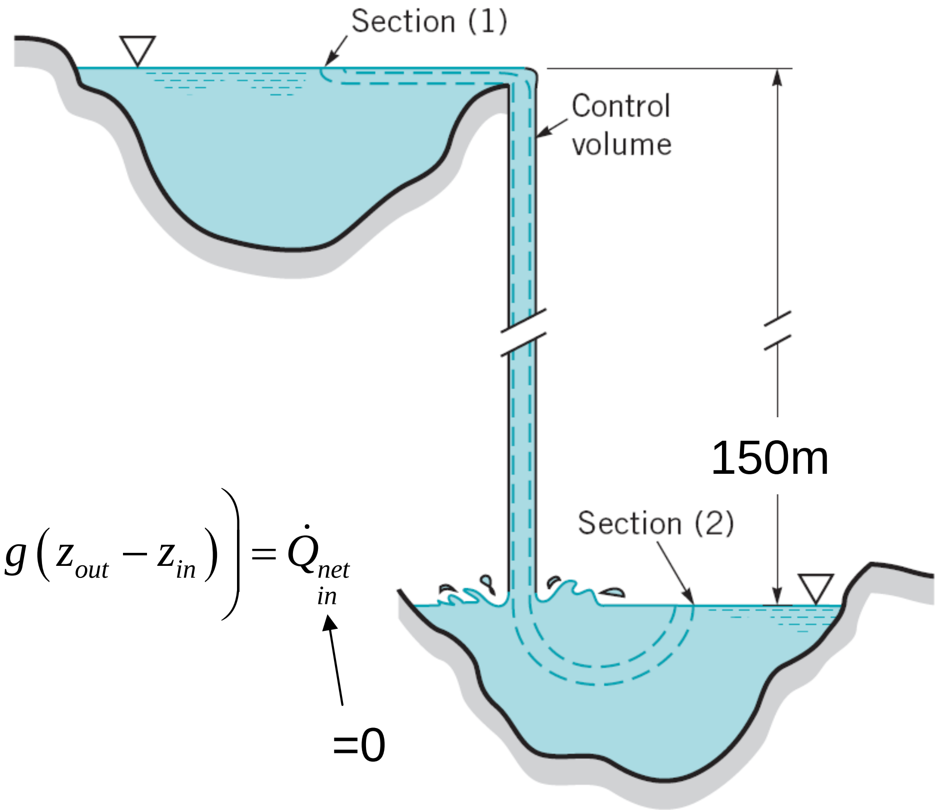
$$\int_{CS}\left(\breve{u}+\frac{p}{\rho}+\frac{V^2}{2}+gz\right)\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA = \left(\breve{u}+\frac{p}{\rho}+\frac{V^2}{2}+gz\right)_{out}\dot{m}_{out} - \left(\breve{u}+\frac{p}{\rho}+\frac{V^2}{2}+gz\right)_{in}\dot{m}_{in}$$

$$\dot{m}\left(\breve{u}_{out}-\breve{u}_{in}+\left(\frac{p}{\rho}\right)_{out}-\left(\frac{p}{\rho}\right)_{in}+\left(\frac{V_{out}{}^2-V_{in}{}^2}{2}\right)+g\left(z_{out}-z_{in}\right)\right) = \dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{\text{shaft}\\\text{net in}}}$$

$$\breve{h} = \breve{u} + \left(\frac{p}{\rho}\right)$$

Example: Temperature change at a water fall

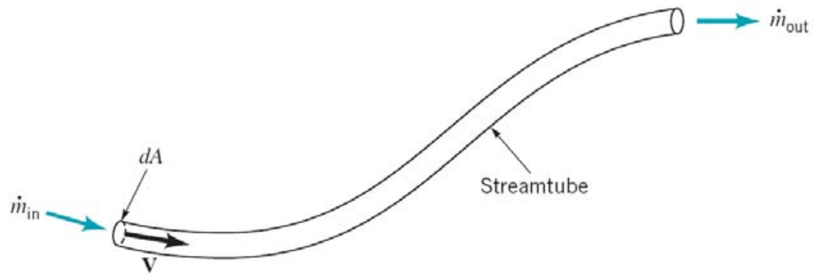
- find the temperature change after a water fall, c_{water}=4.19 kJ/kg·K



$$\dot{m}\left(\breve{u}_{out}-\breve{u}_{in}+\underbrace{\left(\frac{p}{\rho}\right)_{out}-\left(\frac{p}{\rho}\right)_{in}}_{=0}+\underbrace{\left(\frac{V_{out}^{\;2}-V_{in}^{\;2}}{2}\right)}_{=0}+g\left(z_{out}-z_{in}\right)\right)=\underbrace{\dot{Q}_{\substack{net\\in}}}_{=0}$$

Energy equation vs Bernoulli equation

- Let's return to our one-stream volume, steady flow (also no shaft power)



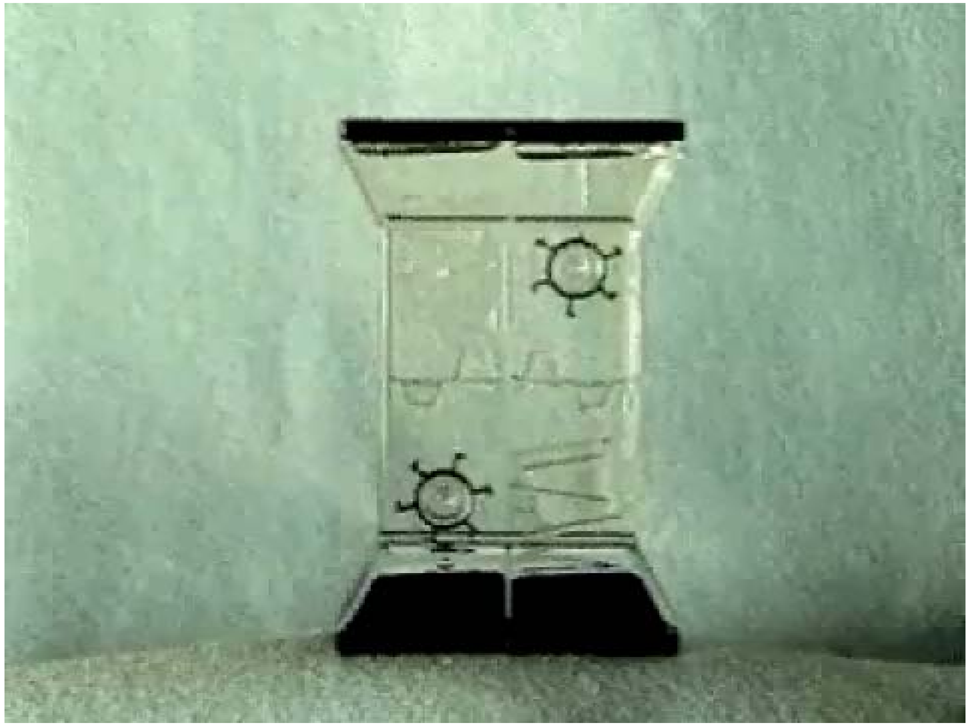
$$\dot{m}\left(\breve{u}_{out}-\breve{u}_{in}+\left(\frac{p}{\rho}\right)_{out}-\left(\frac{p}{\rho}\right)_{in}+\left(\frac{V_{out}^{2}-V_{in}^{2}}{2}\right)+g\left(z_{out}-z_{in}\right)\right)=\dot{Q}_{\substack{net\\in}}$$

$$\underbrace{\frac{p_{out}}{\rho}+\frac{V_{out}^{2}}{2}+gz_{out}}_{\textit{available energy}}=\frac{p_{in}}{\rho}+\frac{V_{in}^{2}}{2}+gz_{in}-\underbrace{(\breve{u}_{out}-\breve{u}_{in}-q_{\substack{net\\in}})}_{\textbf{loss}},\qquad q_{\substack{net\\in}}=\frac{\dot{Q}_{\substack{net\\in}}}{\dot{m}}$$

- Comparing with Bernoulli equation: $\breve{u}_{out}-\breve{u}_{in}-\dot{q}_{\substack{net\\in}}=0$

i.e. steady incompressible flow should be also ***frictionless***

Energy transfer

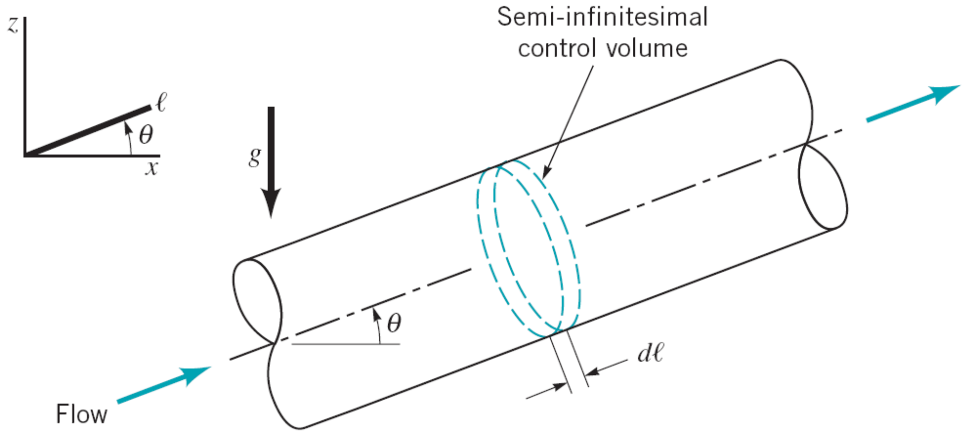


Work must be done on the device shown to turn it over because the system gains potential energy as the heavy (dark) liquid is raised above the light (clear) liquid. This potential energy is converted into kinetic energy which is either dissipated due to friction as the fluid flows down the ramp or is converted into power by the turbine and then dissipated by friction. The fluid finally becomes stationary again. The initial work done in turning it over eventually results in a very slight increase in the system temperature

Second law of thermodynamics

- Let's apply "stream line energy equation" to an infinitesimally thin volume

$$\dot{m}\left[d\breve{u}+d\left(\frac{p}{\rho}\right)+d\left(\frac{V^2}{2}\right)+gdz\right]=\delta\dot{Q}_{\substack{net\\in}}$$



For closed system in the absence of additional work:

$$dU=TdS-pdV \quad \Rightarrow \quad d\breve{u}=Tds-pd\left(\frac{1}{\rho}\right)$$

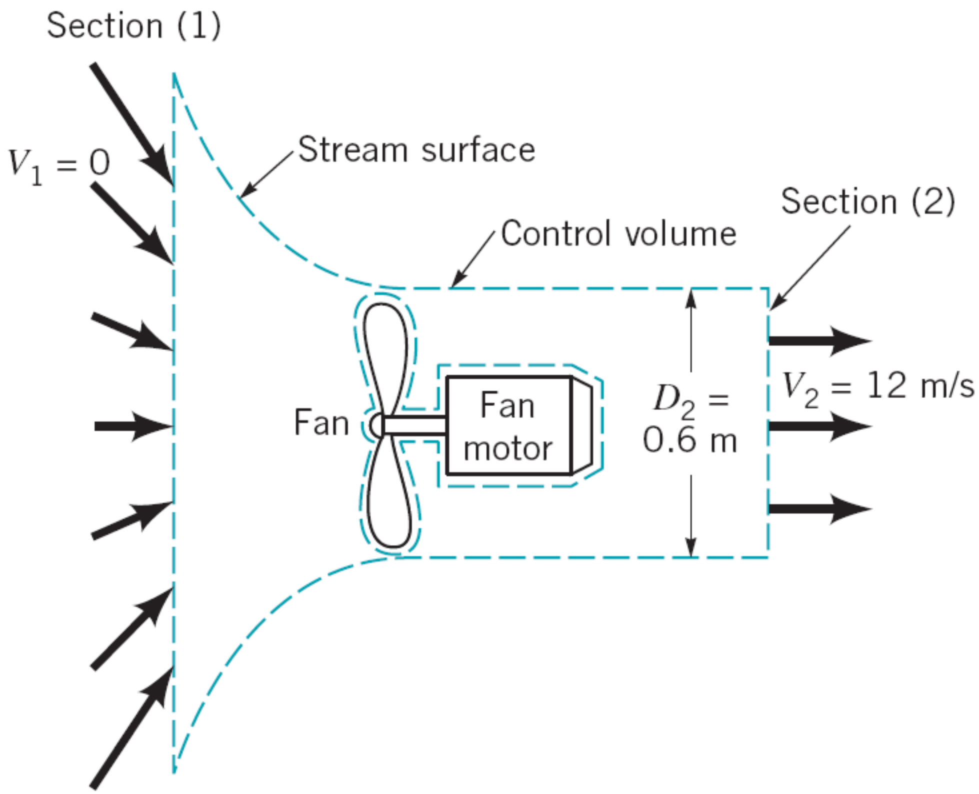
$$\left[\frac{dp}{\rho}+d\left(\frac{V^2}{2}\right)+gdz\right]=-(Tds-\delta\dot{q}_{\substack{net\\in}})$$

If we take into account Clausius inequality: $dS-\frac{dq}{T}\geq 0$

$$-\left[\frac{dp}{\rho}+d\left(\frac{V^2}{2}\right)+gdz\right]\geq 0$$

Example: Fan Efficiency

- An axial-flow ventilating fan is driven by the 0.4kW motor and producing 12m/s speed in a 0.6m diameter channel. Determine the useful effect and efficiency. Air density is 1.23 kg/m³.



$$w_{\substack{\text{shaft}\\ \text{net in}}} - loss = \left(\frac{p_{out}}{\rho} + \frac{V_{out}^{\ 2}}{2} + gz_{out} \right) - \left(\frac{p_{in}}{\rho} + \frac{V_{in}^{\ 2}}{2} + gz_{in} \right)$$

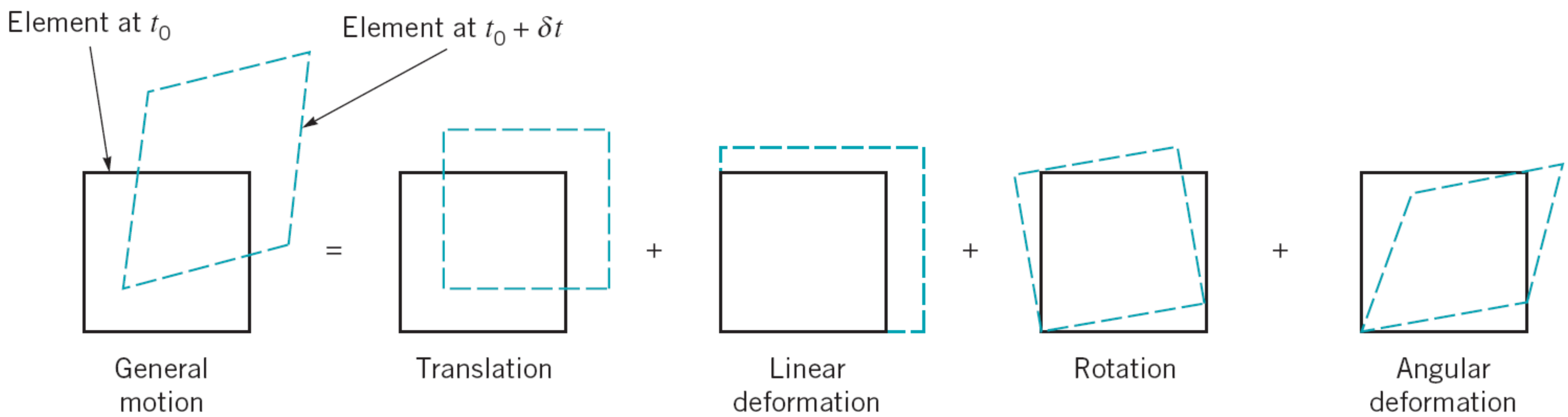
$$w_{\substack{\text{shaft}\\ \text{net in}}} = \frac{W_{\substack{\text{shaft}\\ \text{net in}}}}{\dot{m}}$$

Differential analysis of Fluid Flow

- The aim: to produce differential equation describing the motion of fluid in detail

Fluid Element Kinematics

- Any fluid element motion can be represented as consisting of translation, linear deformation, rotation and angular deformation



Velocity and acceleration field

- Velocity field

$$\boldsymbol{V} = u\hat{\boldsymbol{i}} + v\hat{\boldsymbol{j}} + w\hat{\boldsymbol{k}}$$

- Acceleration

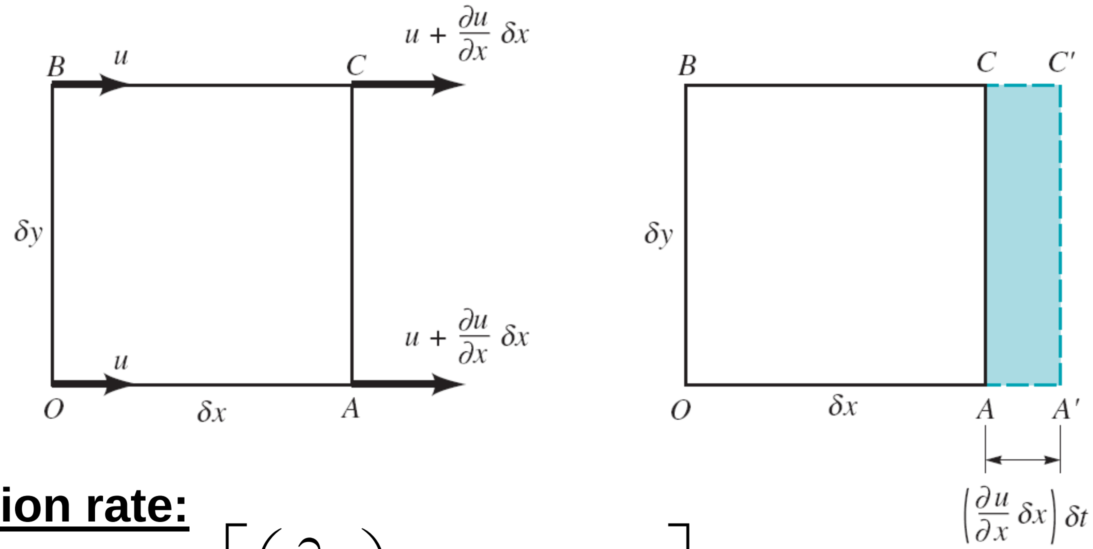
$$\mathbf{a}(r,t) = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = \frac{D\mathbf{V}}{Dt}$$

- Material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + (\mathbf{V} \bullet \nabla)$$

Linear motion and deformation

- Let's consider stretching of a fluid element under velocity gradient in one direction

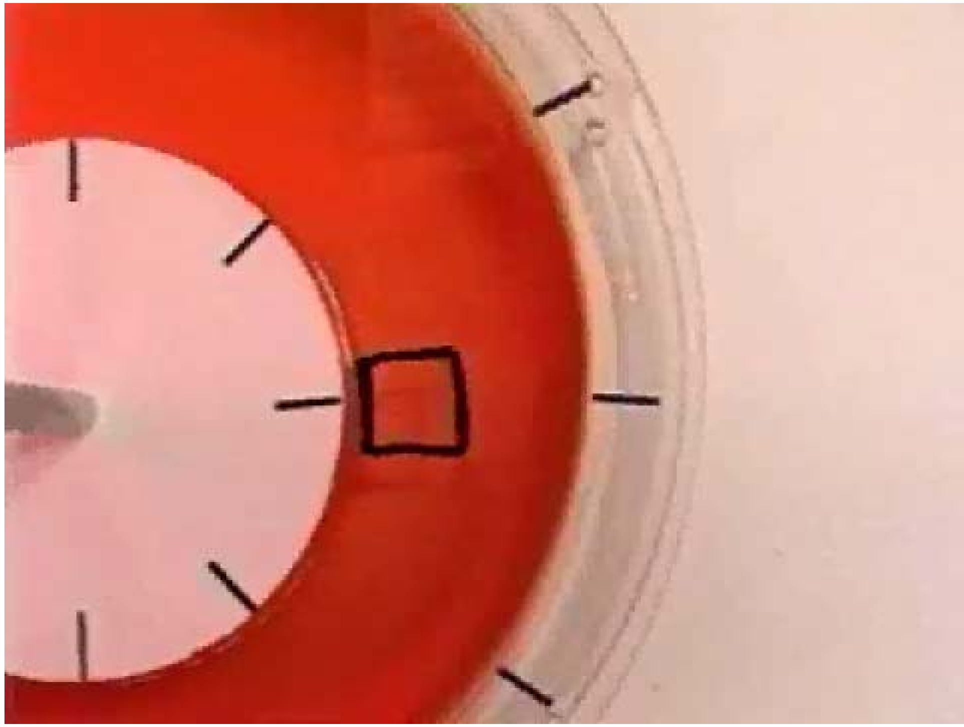


<u>**Volumetric dilatation rate:**</u>

$$\frac{1}{\delta \forall}\frac{d(\delta \forall)}{dt} = \lim_{\delta t \to 0}\left[\frac{\left(\frac{\partial u}{\partial x}\right)\delta x \delta t\, \delta y \delta z}{\delta x \delta y \delta z\, \delta t}\right] = \left(\frac{\partial u}{\partial x}\right)$$

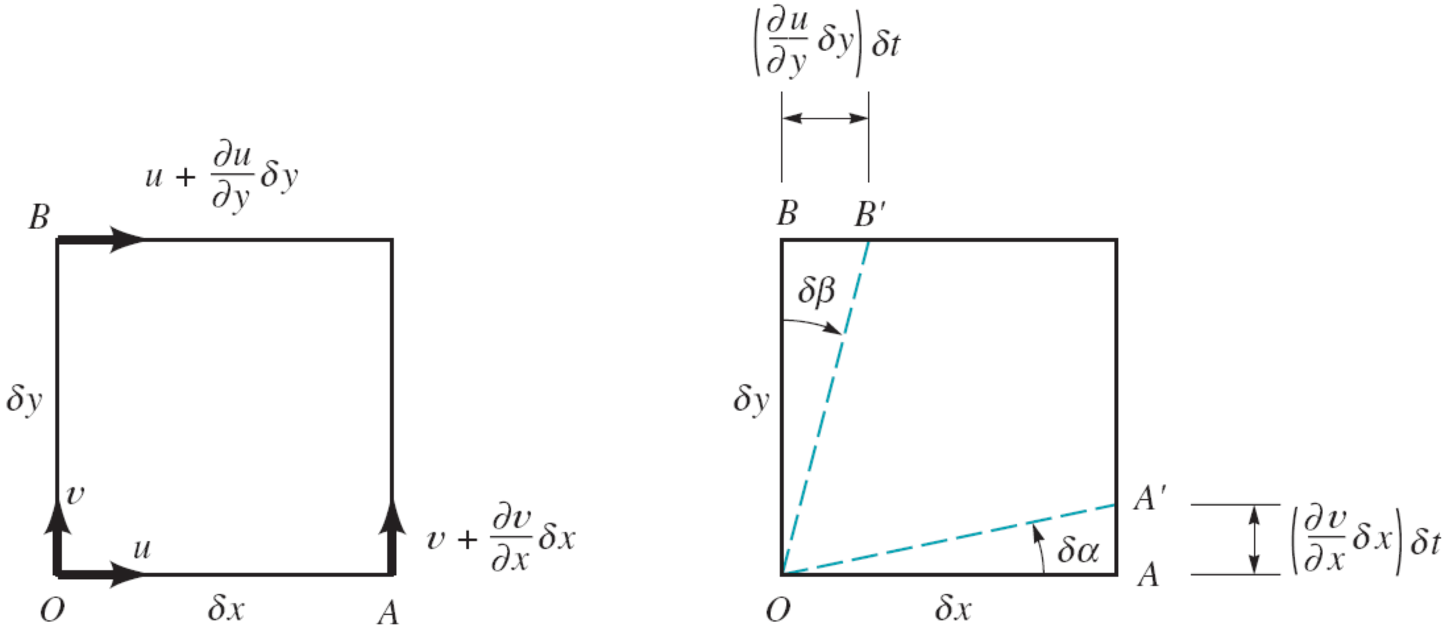
$$\frac{1}{\delta \forall}\frac{d(\delta \forall)}{dt} = \left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial z}\right) = \nabla \cdot \boldsymbol{V}$$

Angular motion and deformation



Fluid elements located in a moving fluid move with the fluid and generally undergo a change in shape (angular deformation).
A small rectangular fluid element is located in the space between concentric cylinders. The inner wall is fixed. As the outer wall moves, the fluid element undergoes an angular deformation. The rate at which the corner angles change (rate of angular deformation) is related to the shear stress causing the deformation

Angular motion and deformation



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$$\omega_{OA} = \lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t} \approx \frac{(\partial v / \partial x)\delta x \delta t}{\delta x}\frac{1}{\delta t} = \frac{\partial v}{\partial x}$$

$$\omega_{OB} = \frac{\partial u}{\partial y}$$

- Rotation is defined as the average of those velocities:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Angular motion and deformation

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\boldsymbol{\omega} = \frac{1}{2}\nabla \times \boldsymbol{V} = \frac{1}{2}\begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} =$$

$$= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\boldsymbol{i}} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\boldsymbol{j}} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\boldsymbol{k}}$$

- Vorticity is defined as twice the rotation vector

$$\zeta = 2\omega = \nabla \times \boldsymbol{V}$$

- If rotation (and vorticity) is zero flow is called **irrotational**

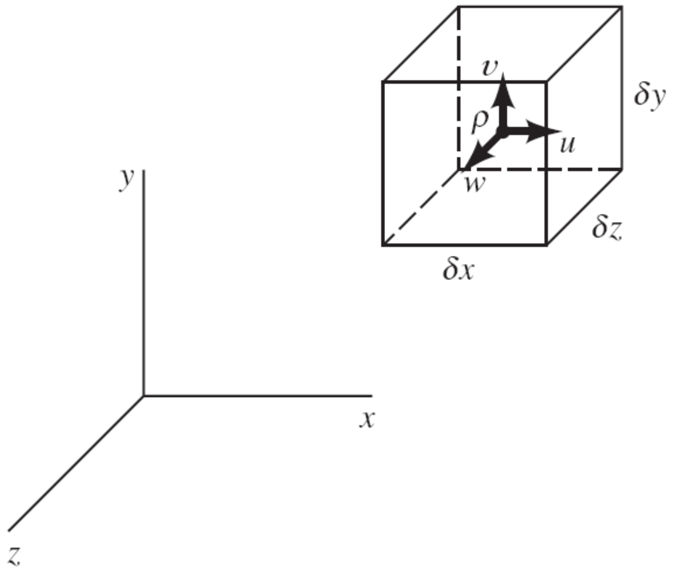
Angular motion and deformation

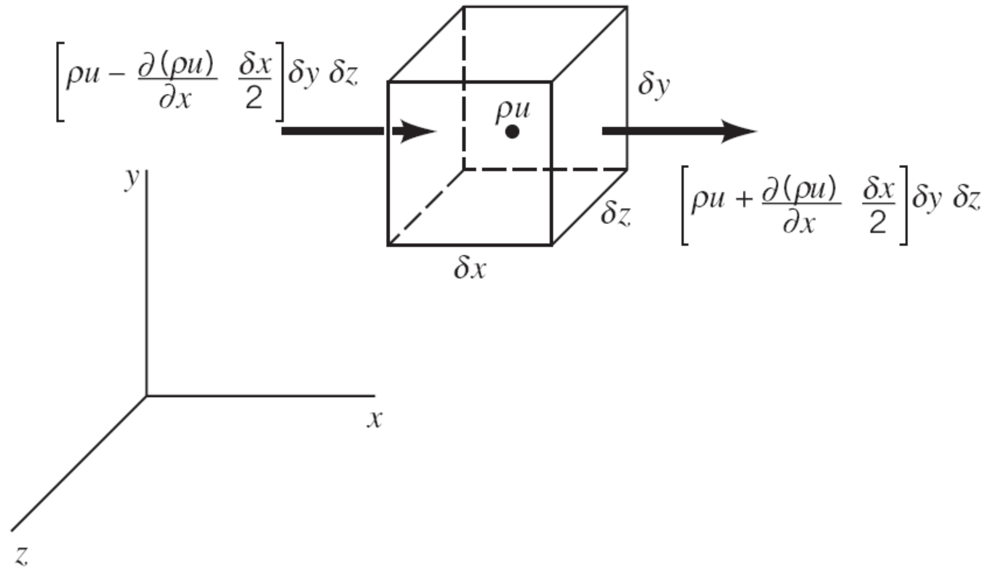
- Rate of shearing strain (or rate of angular deformation) can be defined as sum of fluid element rotations:

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Conservation of mass

- As we found before: $$\frac{DM_{Sys}}{Dt} = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CV} \rho \mathbf{V}\cdot\hat{\mathbf{n}}dA = 0$$





$$\frac{\partial}{\partial t}\int_{CV} \rho d\forall = \frac{\partial \rho}{\partial t}\delta x\delta y\delta z$$

Flow of mass in x-direction: $$\frac{\partial(\rho u)}{\partial x}\delta x\delta y\delta z$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad \frac{\partial \rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0$$

Conservation of mass

- Incompressible flow

$$\nabla \cdot \boldsymbol{V} = 0 \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- Flow in cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r \rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

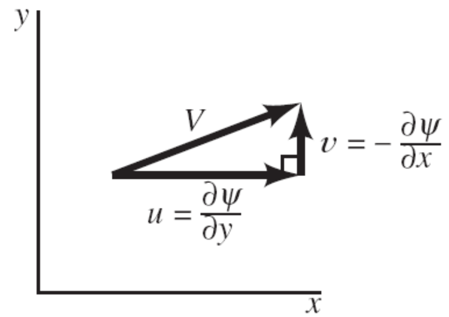
- Incompressible flow in cylindrical coordinates

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (v_\theta)}{\partial \theta} + \frac{\partial (v_z)}{\partial z} = 0$$

Stream function

- 2D incompressible flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



- We can define a scalar function such that

$$u = \frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \psi}{\partial x}$$

Stream function

- Lines along which stream function is const are stream lines:

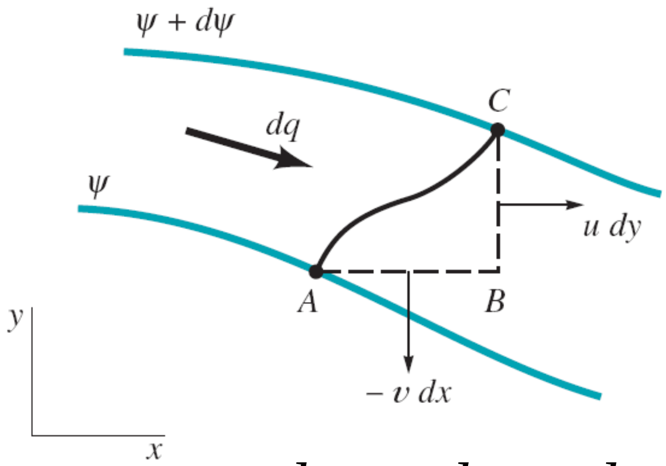
$$\frac{dy}{dx} = \frac{v}{u}$$

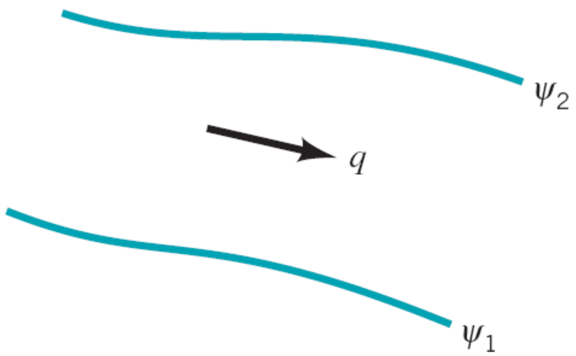
Indeed:

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -vdx + udy = 0$$

Stream function

- Flow between streamlines





$$dq = udy - vdx$$

(b)

$$dq = \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = d\psi$$

$$q = \psi_2 - \psi_1$$

Description of forces

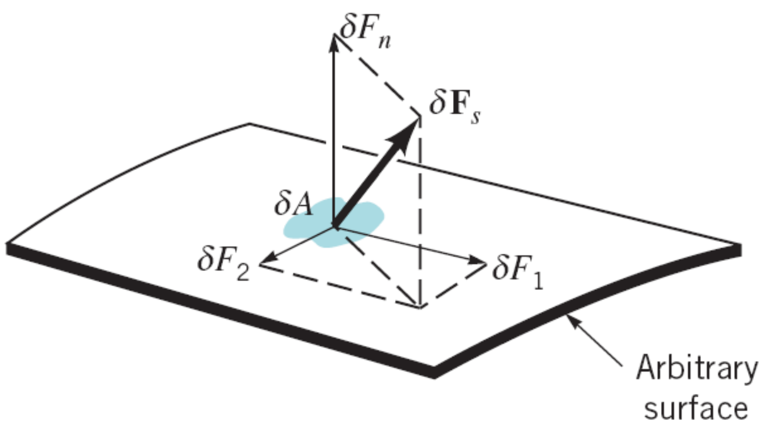
Forces

Body forces – distributed through the element, e.g. Gravity

$$\delta \boldsymbol{F} = \delta m \boldsymbol{g}$$

Surface forces – result of interaction with the surrounding elements: e.g. Stress

Normal stress



Shearing stresses

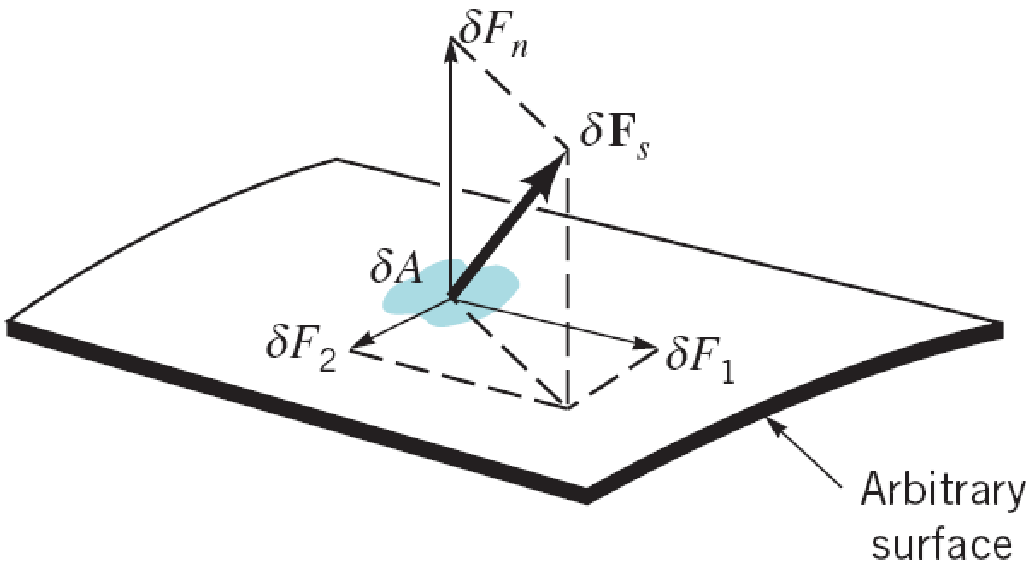
Linear forces: Surface tension

Stress acting on a fluidic element

- normal stress $$\sigma_n = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A}$$

- shearing stresses

$$\tau_1 = \lim_{\delta A \to 0} \frac{\delta F_1}{\delta A} \qquad \tau_2 = \lim_{\delta A \to 0} \frac{\delta F_2}{\delta A}$$



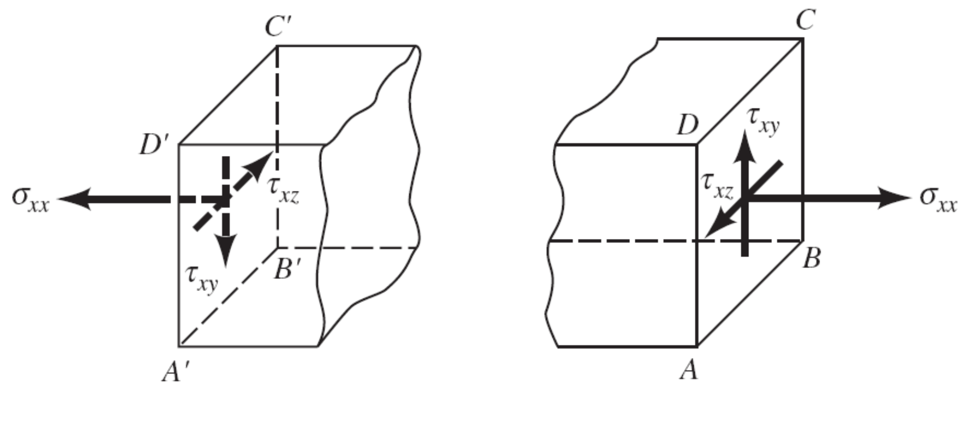
Stresses: double subscript notation

- normal stress: σ_{xx}

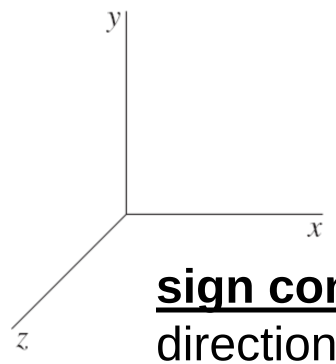
- shearing stress: τ_{xy} τ_{xz}

normal to the plane

direction of stress



(b) (a)



sign convention: positive stress is directed in positive axis directions if surface normal is pointing in the positive direction

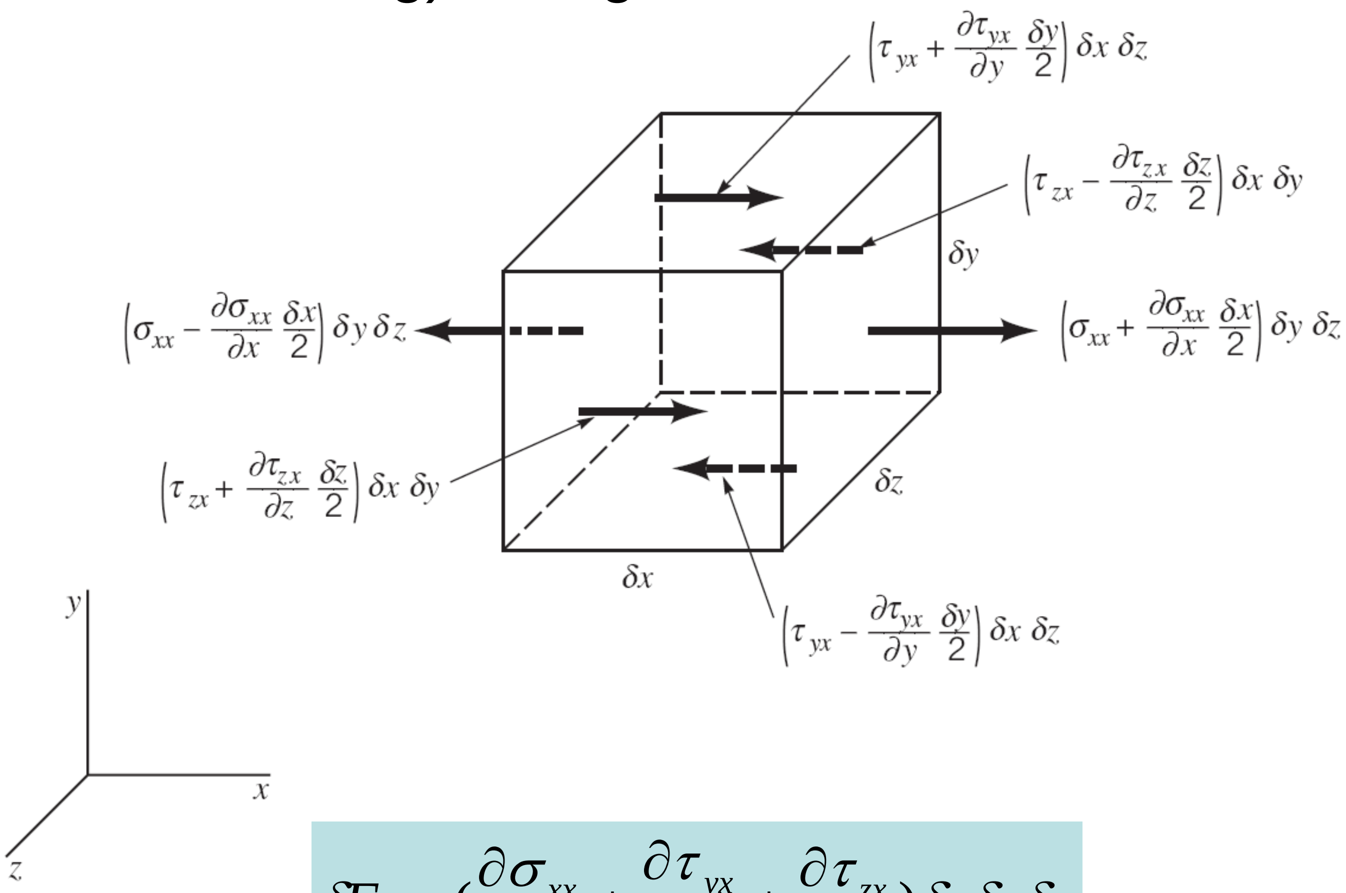
Stress tensor

- To define stress at a point we need to define “stress vector” for all 3 perpendicular planes passing through the point

$$\tau = \begin{pmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{pmatrix}$$

Force on a fluid element

- To find force in each direction we need to sum all forces (normal and shearing) acting in the same direction



$$\delta F_x = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x \delta y \delta z$$

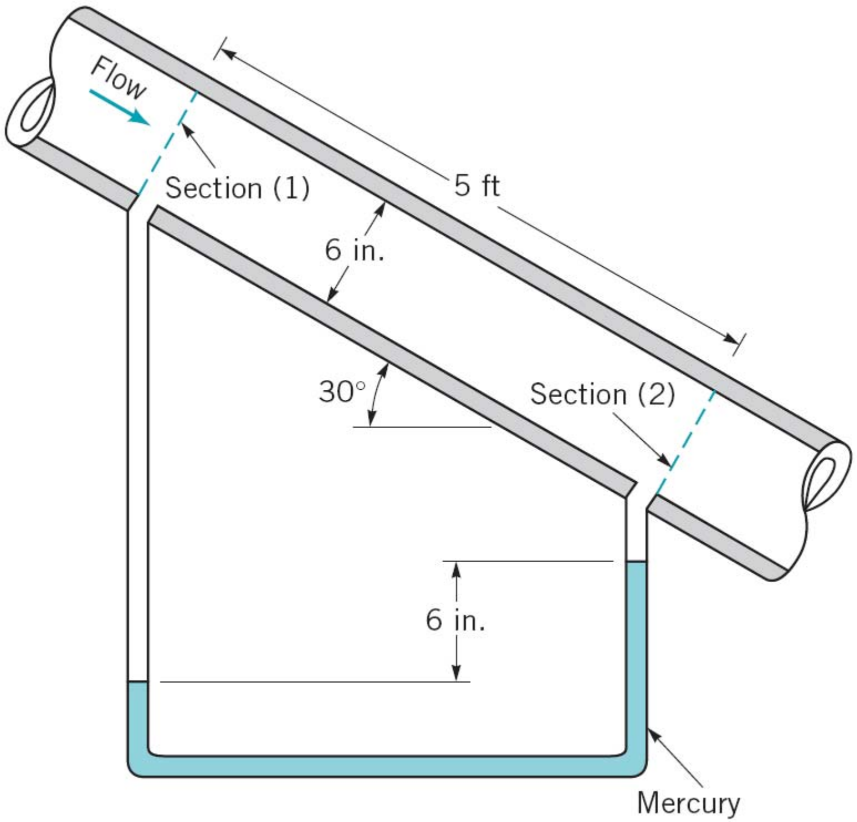
Problems

- **6.2** A certain flow field is given by equation:

$$\vec{V} = (3x^2 + 1)\vec{i} - 6xy\vec{j}$$

Determine expression for local and convective components of the acceleration in x and y directions

- **5.102** Water flows steadily down the inclined pump. Determine:
  - The pressure difference, p_1-p_2;
  - The loss between sections 1 and 2
  - The axial force exerted on the pipe by water



Problems

- **6.4** The components of the velocity in the flow field are given by:

$$u = x^2 + y^2 + z^2$$

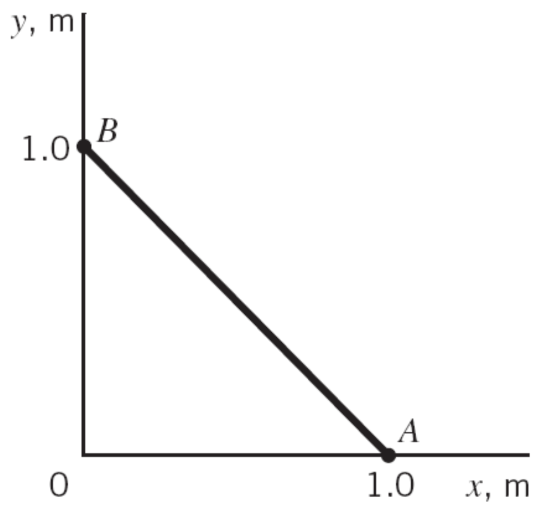
$$v = xy + yz + z^2$$

$$w = -3xz - z^2/2 + 4$$

- Determine the volumetric dilation rate and interpret the results.
- Determine the expression for the rotation vector. Is the flow irrotational?

- **6.22** The stream function for an incompressible flow

$$\psi = 3x^2 y - y^3$$



sketch the streamline passing through the origin; determine of flow across the strait path AB