Pressure in stationary and moving fluid

Lab-On-Chip: Lecture 2

Fluid Statics

- **No shearing stress**
- .no relative movement between adjacent fluid particles, i.e. static or moving as a single block

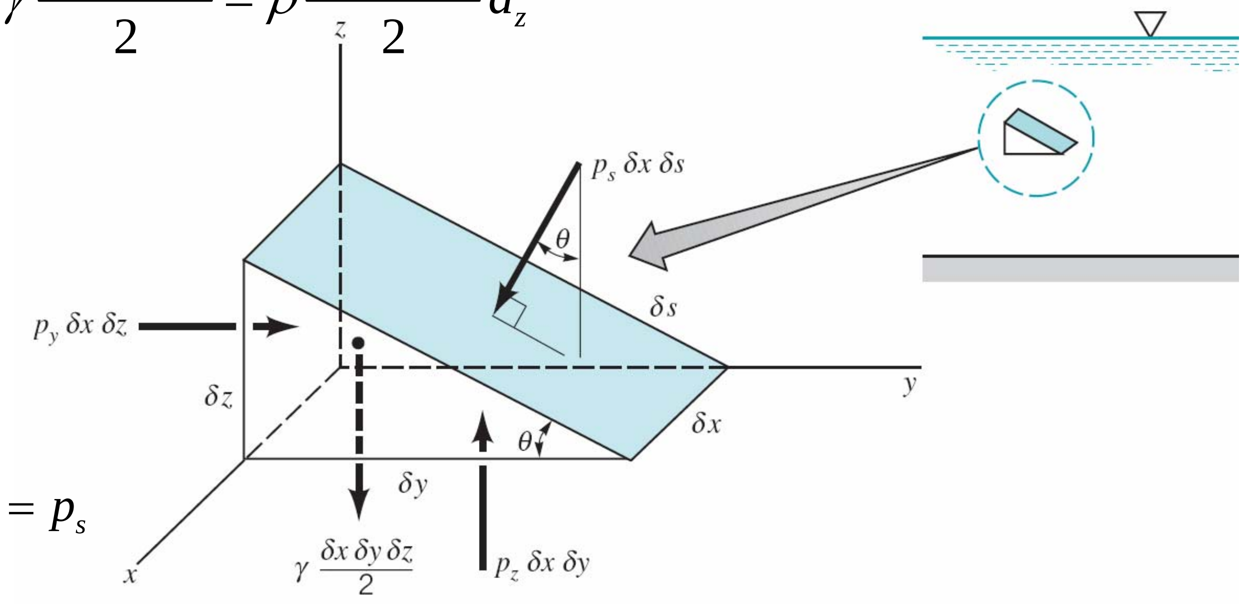
Pressure at a point

Question: How pressure depends on the orientation of the plane ?

Newton’s second law:

$$\sum F_y = p_y \delta x \delta z - p_s \delta x \delta s \sin\theta = \rho \frac{\delta x \delta y \delta z}{2} a_y$$

$$\sum F_z = p_z \delta x \delta y - p_s \delta x \delta s \cos\theta - \gamma \frac{\delta x \delta y \delta z}{2} = \rho \frac{\delta x \delta y \delta z}{2} a_z$$



$$\delta y = \delta s \cos\theta \quad \delta z = \delta s \sin\theta$$

$$p_y - p_s = pa_y \frac{\delta y}{2}$$

$$p_z - p_s = (pa_z + \rho g)\frac{\delta z}{2}$$

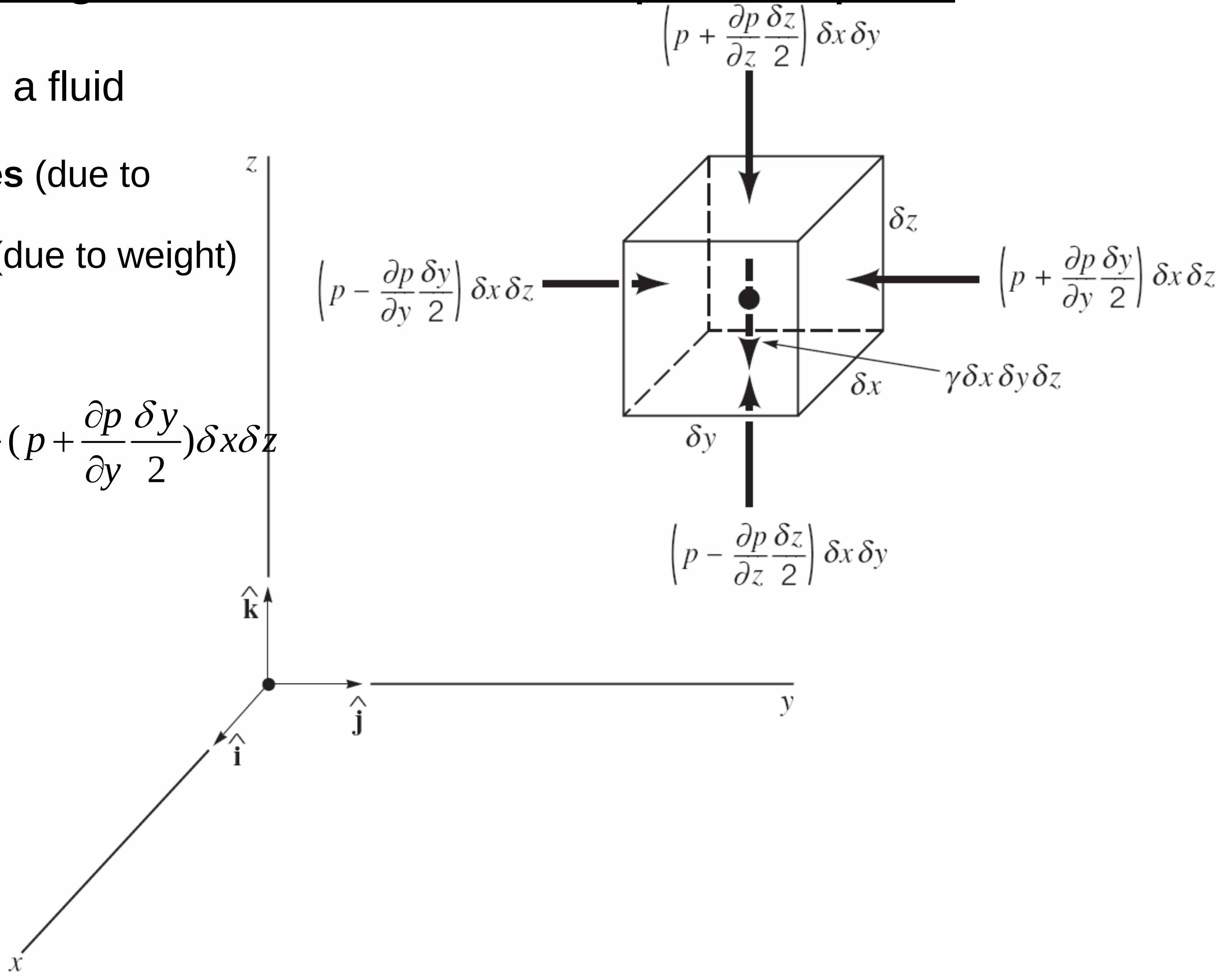
$$if \quad \delta x, \delta y, \delta z \rightarrow 0, \;\; p_y = p_s, p_z = p_s$$

- **Pascal’s law:** pressure doesn’t depend on the orientation of plate (i.e. a scalar number) as long as there are no shearing stresses

Basic equation for pressure field

Question: What is the pressure distribution in liquid in absence shearing stress variation from point to point

- Forces acting on a fluid element::
  - **Surface forces** (due to pressure)
  - **Body forces** (due to weight)



Surface forces:

$$\delta F_y = (p - \frac{\partial p}{\partial y}\frac{\delta y}{2})\delta x \delta z - (p + \frac{\partial p}{\partial y}\frac{\delta y}{2})\delta x \delta z$$

$$\begin{cases} \delta F_y = -\dfrac{\partial p}{\partial y}\delta y \delta x \delta z \\ \delta F_x = -\dfrac{\partial p}{\partial x}\delta y \delta x \delta z \\ \delta F_z = -\dfrac{\partial p}{\partial z}\delta y \delta x \delta z \end{cases}$$

Basic equation for pressure field

Resulting surface force in vector form: $\delta\vec{F} = -(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k})\delta y \delta x \delta z$

If we define a gradient as: $\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k}$ $\qquad \frac{\delta\vec{F}}{\delta y \delta x \delta z} = -\nabla p$

The weight of element is: $-\delta W\vec{k} = -\rho g\, \delta y \delta x \delta z\, \vec{k}$

Newton's second law:

$$\delta\vec{F} - \delta W\vec{k} = \delta m\vec{a}$$

$$-\nabla p \delta y \delta x \delta z - \rho g\, \delta y \delta x \delta z\, \vec{k} = -\rho g\, \delta y \delta x \delta z\, \vec{a}$$

General equation of motion for a fluid w/o shearing stresses

$$\boxed{-\nabla p - \rho g\vec{k} = -\rho g\vec{a}}$$

Pressure variation in a fluid at rest

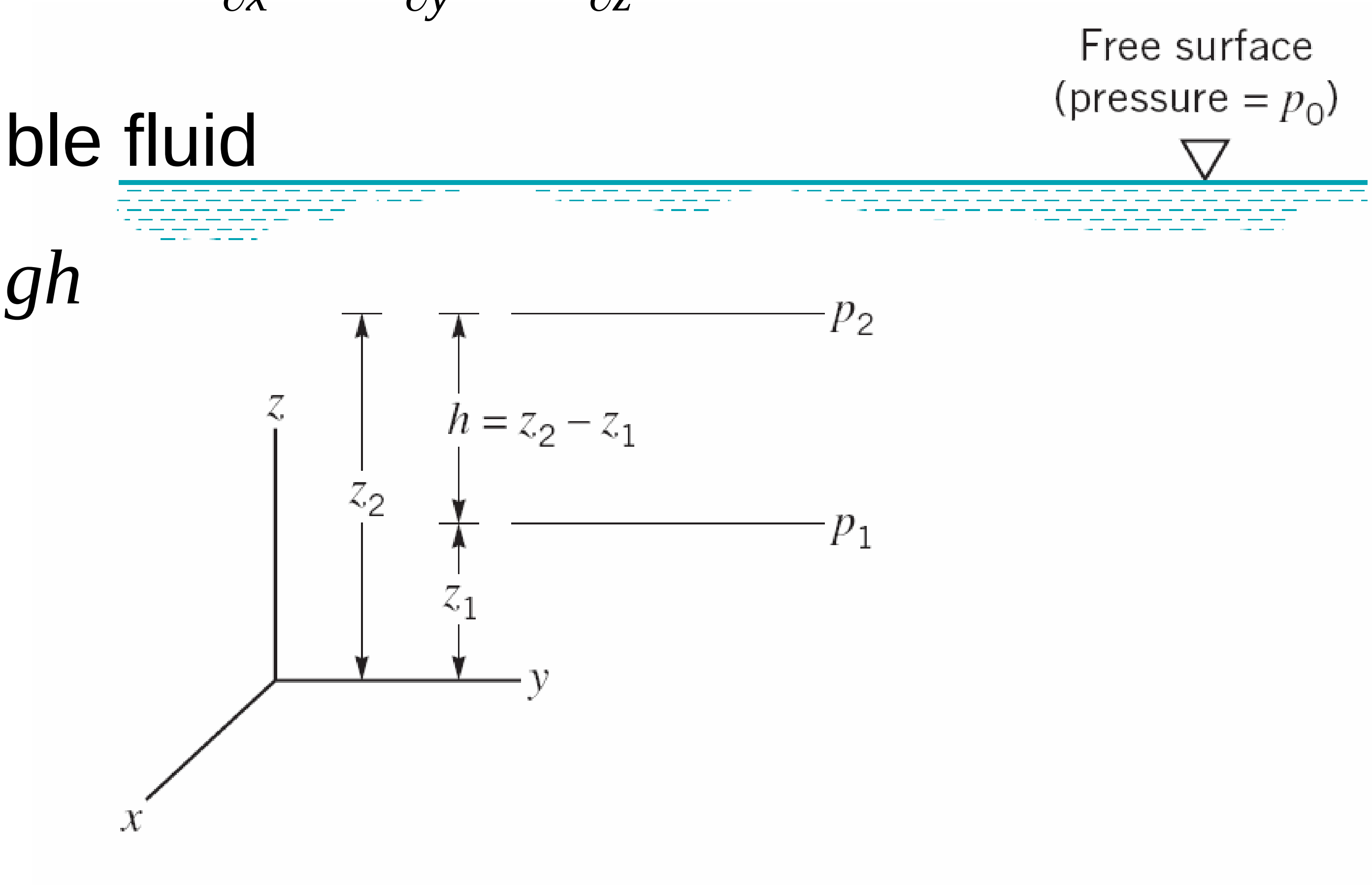
- At rest a=0

$$-\nabla p - \rho g\vec{k} = 0$$

$$\frac{\partial p}{\partial x} = 0 \quad \frac{\partial p}{\partial y} = 0 \quad \frac{\partial p}{\partial z} = -\rho g$$

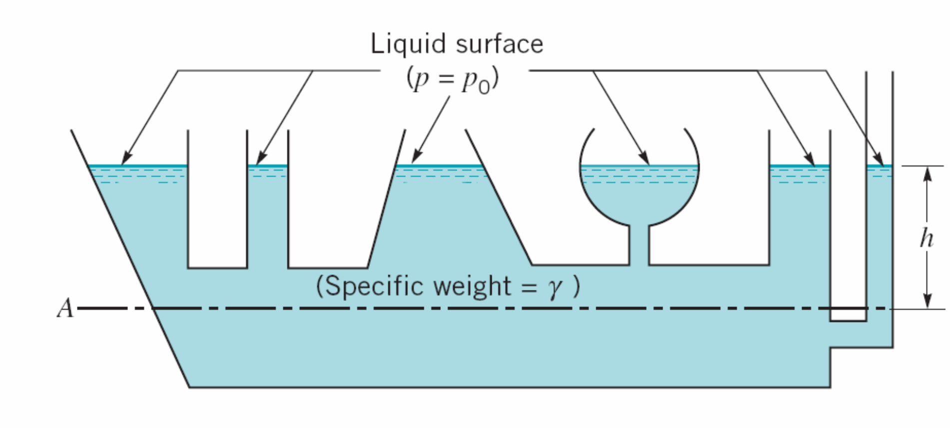
- Incompressible fluid

$$p_1 = p_2 + \rho g h$$

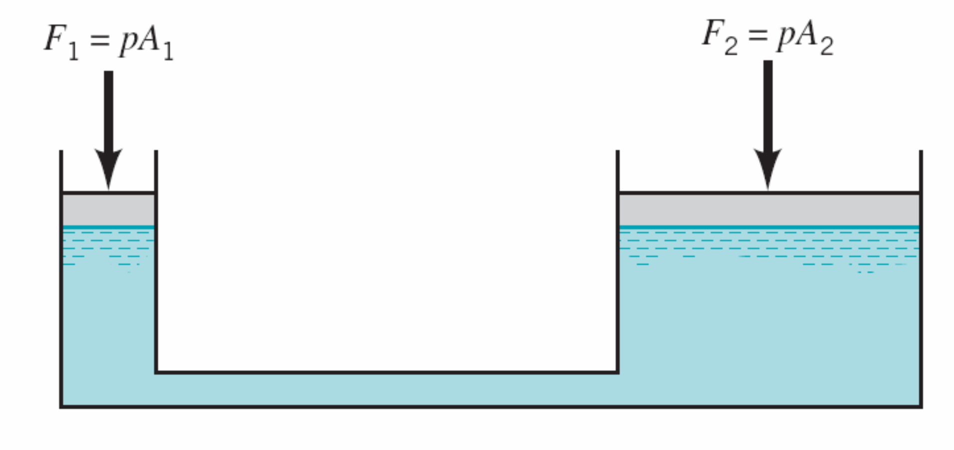


Fluid statics

Same pressure –
much higher force!



Fluid equilibrium



Transmission of fluid pressure, e.g. in hydraulic lifts

- Pressure depends on the depth in the solution not on the lateral coordinate

Compressible fluid

- Example: let's check pressure variation in the air (in atmosphere) due to compressibility:
  - Much lighter than water, 1.225 kg/m^3 against 1000kg/m^3 for water
  - Pressure variation for small height variation are negligible
  - For large height variation compressibility should be taken into account:

$$p = \frac{nRT}{V} = \rho RT$$

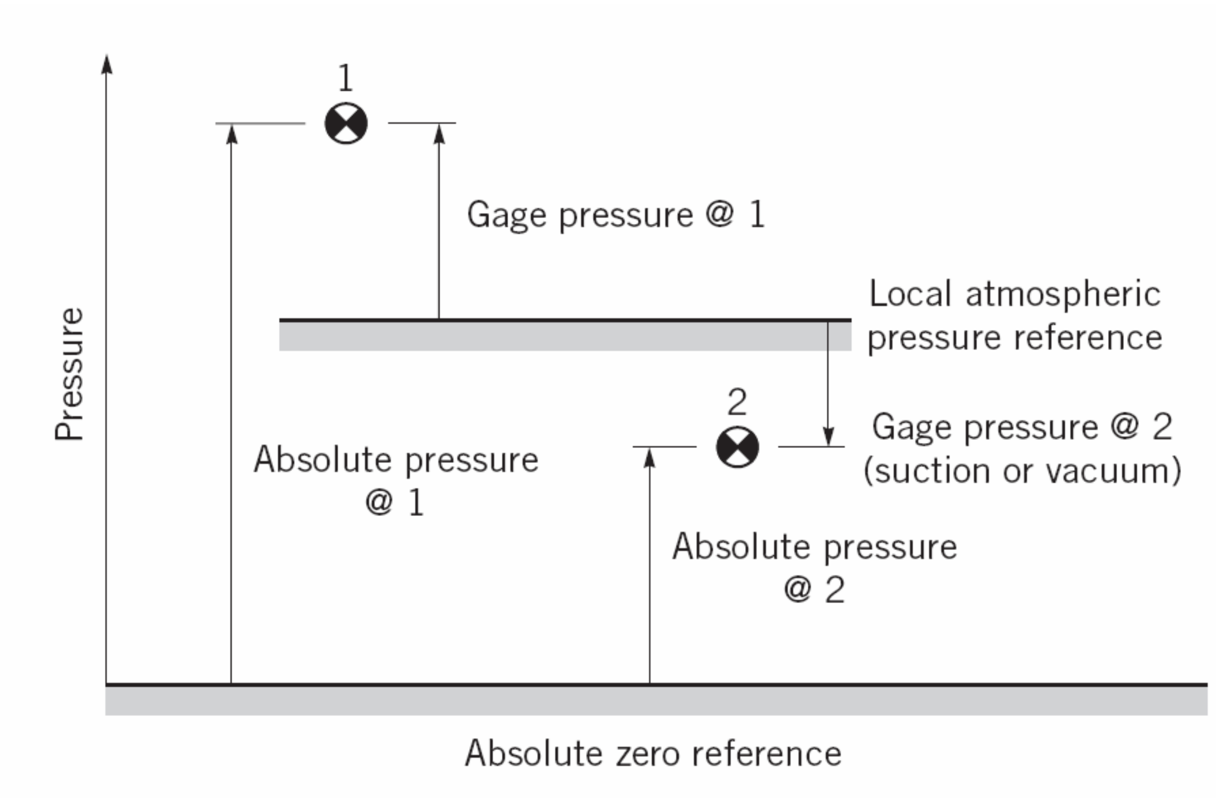
$$\frac{dp}{dz} = -\rho g = -\frac{gp}{RT}$$

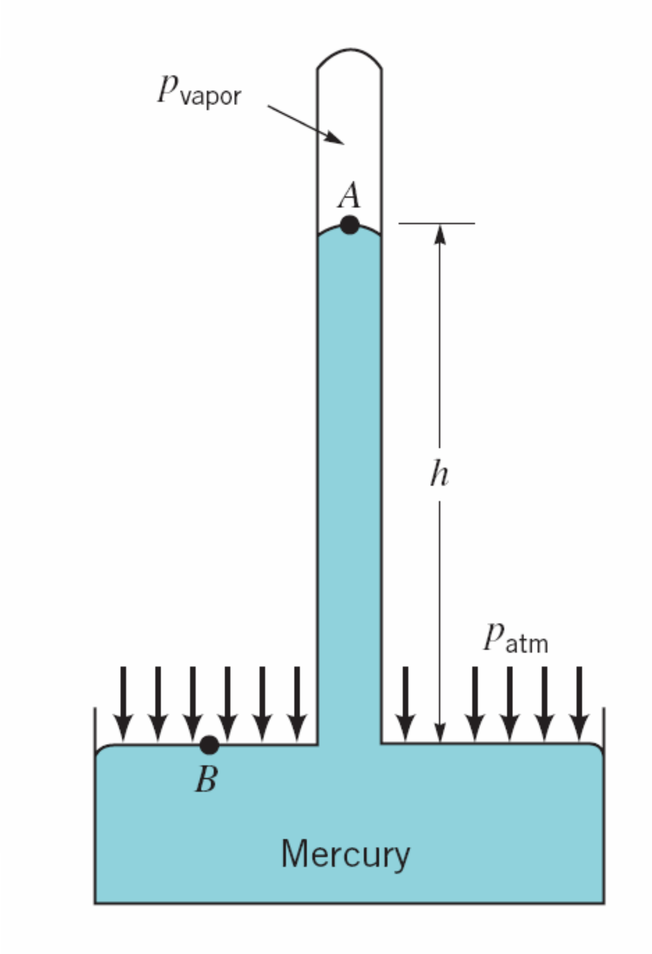
~8 km

$$\text{assuming } T = const \Rightarrow p_2 = p_1 \exp\left[\frac{g(z_1 - z_2)}{RT_0}\right];\ p(h) = p_0 e^{-h/H}$$

Measurement of pressure

- Pressures can be designated as **absolute** or **gage (gauge) pressures**



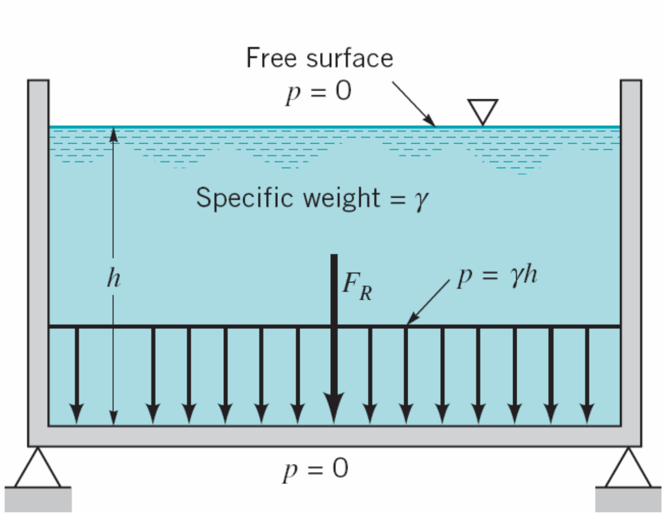


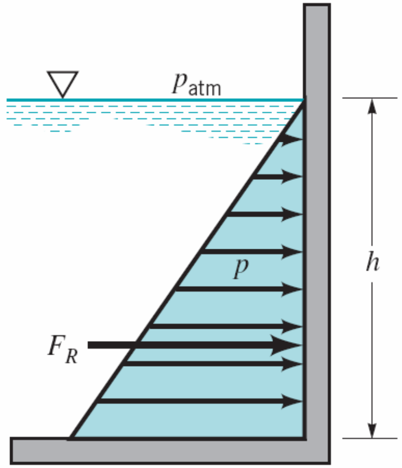
$$p_{atm} = \gamma h + p_{vapor}$$

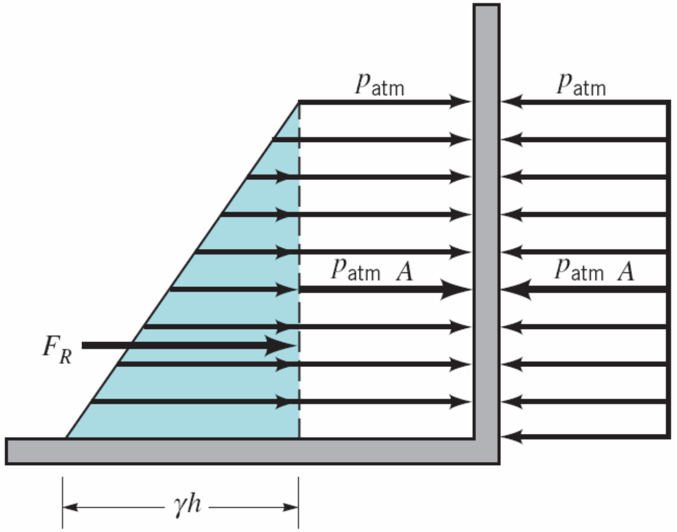
very small!

Hydrostatic force on a plane surface

- For fluid in rest, there are no shearing stresses present and the force must be **perpendicular** to the surface.
- Air pressure acts on both sides of the wall and will cancel.







Force acting on a side wall in rectangular container: $F_R = p_{av} A = \rho g \frac{h}{2} bh$

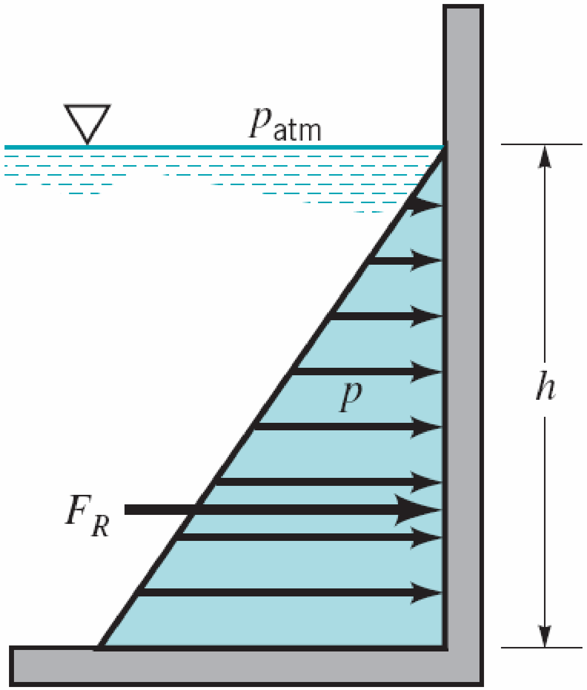
Example: Pressure force and moment acting on aquarium walls

- Force acting on the wall

$$F_R = \int_A \rho gh\, dA = \int_0^H \rho g \left(H - y\right) \cdot bdy = \rho g \frac{H^2}{2} b$$

- Generally: $F_R = \rho g \sin\theta \int_A ydA = \rho g \sin\theta\, y_c A$

Centroid (first moment of the area)



- Momentum of force acting on the wall

$$F_R y_R = \int_A \rho gh\, ydA = \int_0^H \rho g \left(H - y\right) y \cdot bdy = \rho g \frac{H^3}{6} b$$

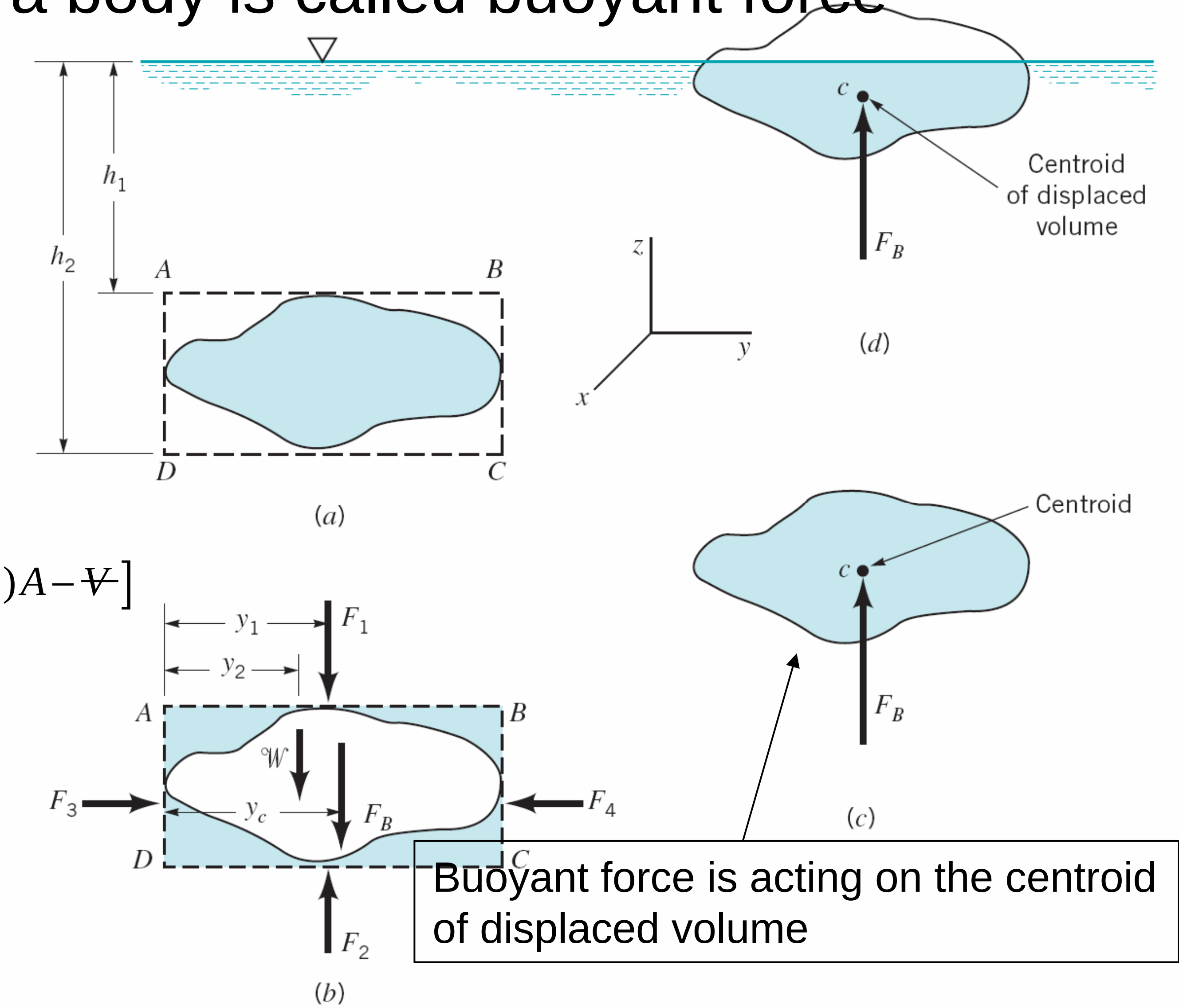
$$y_R = H/3$$

- Generally, $y_R = \dfrac{\int_A y^2 dA}{y_c A}$

Pressure force on a curved surface

Buoyant force: Archimedes principle

- when a body is totally or partially submerged a fluid force acting on a body is called buoyant force



$$F_B = F_2 - F_1 - W$$

$$F_2 - F_1 = \rho g(h_2 - h_1)A$$

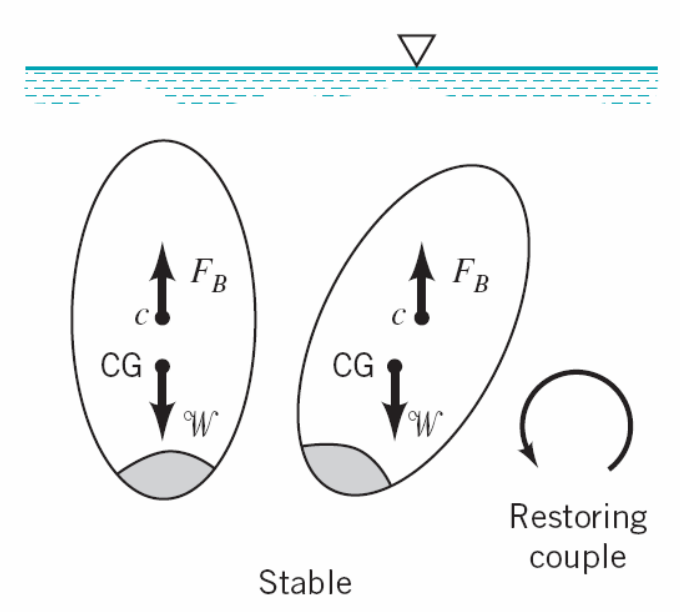
$$F_B = \rho g(h_2 - h_1)A - \rho g\left[(h_2 - h_1)A - \mathcal{V}\right]$$

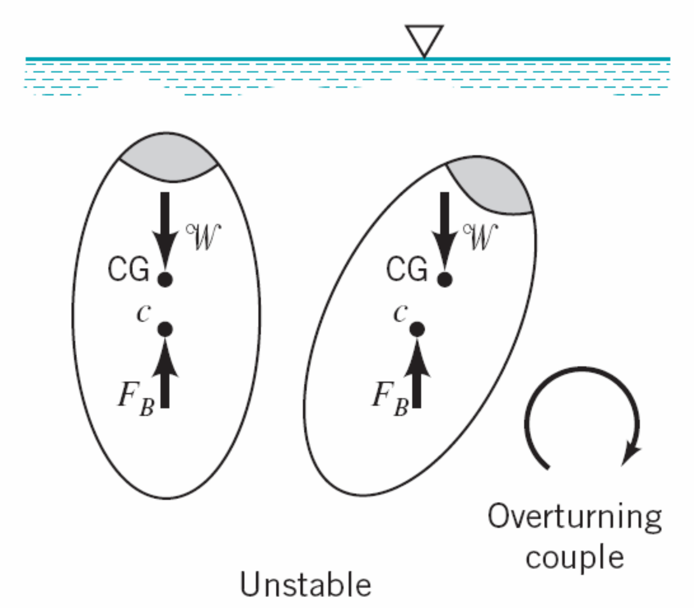
$$\boxed{F_B = \rho g \mathcal{V}}$$

Buoyant force is acting on the centroid of displaced volume

Stability of immersed bodies

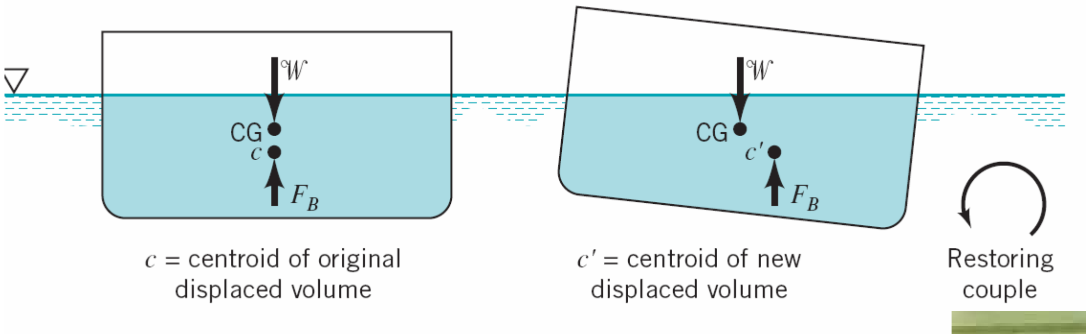
- Totally immersed body



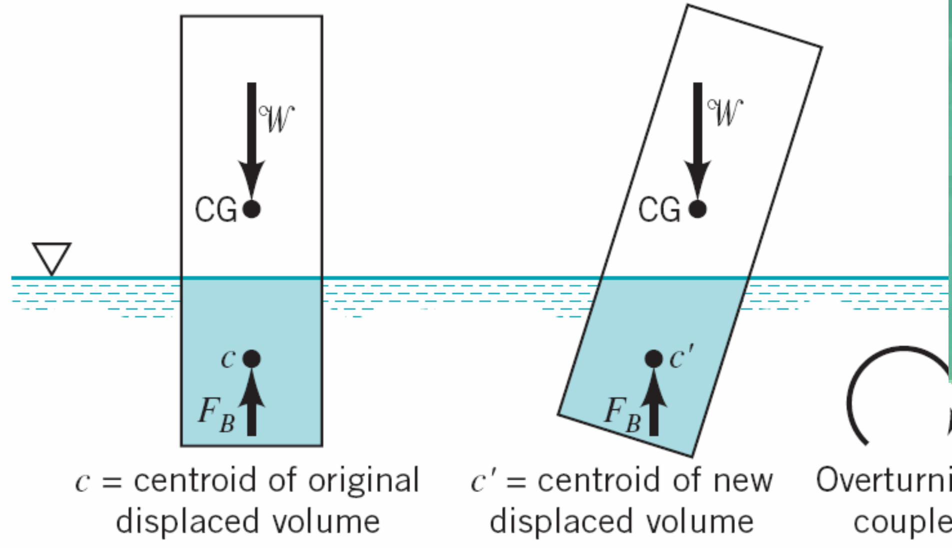


Stability of immersed bodies

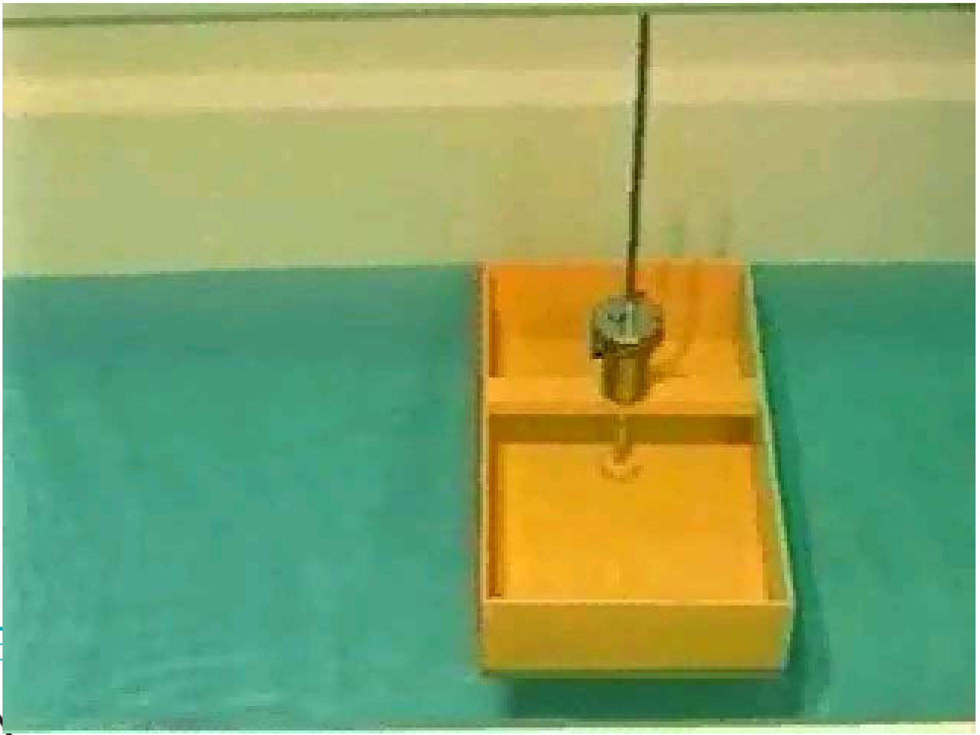
- Floating body



Stable



Unstable

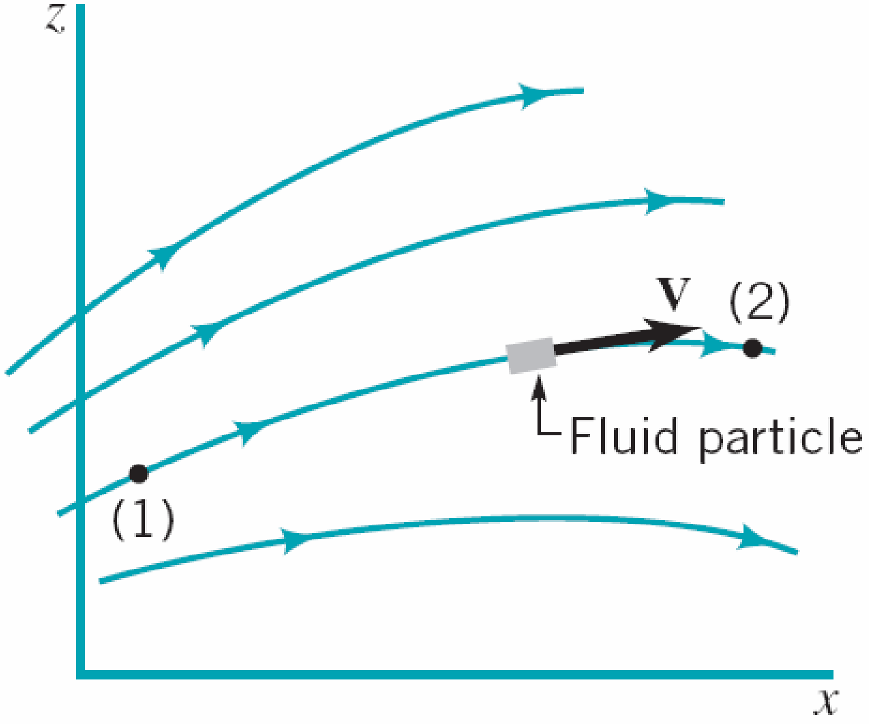


Elementary fluid dynamics: Bernoulli equation

Bernuolli equation – ”the most used and most abused equation in fluid mechanics”

Assumptions:

- steady flow: each fluid particle that passes through a given point will follow a the same path
- inviscid liquid (no viscosity, therefore no thermal conductivity

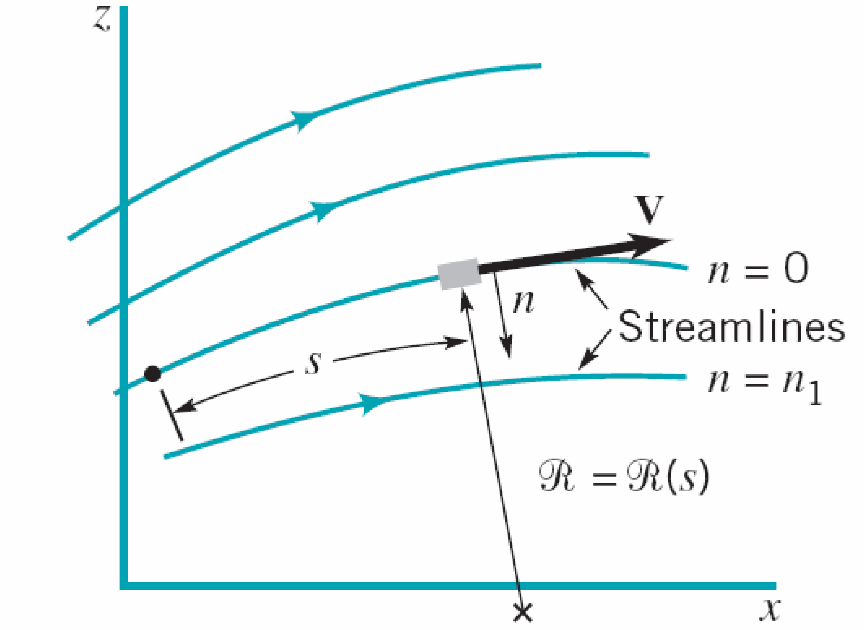


F= ma

Net pressure force + Net gravity force

Streamlines

Streamlines: the lines that are tangent to velocity vector through the flow field

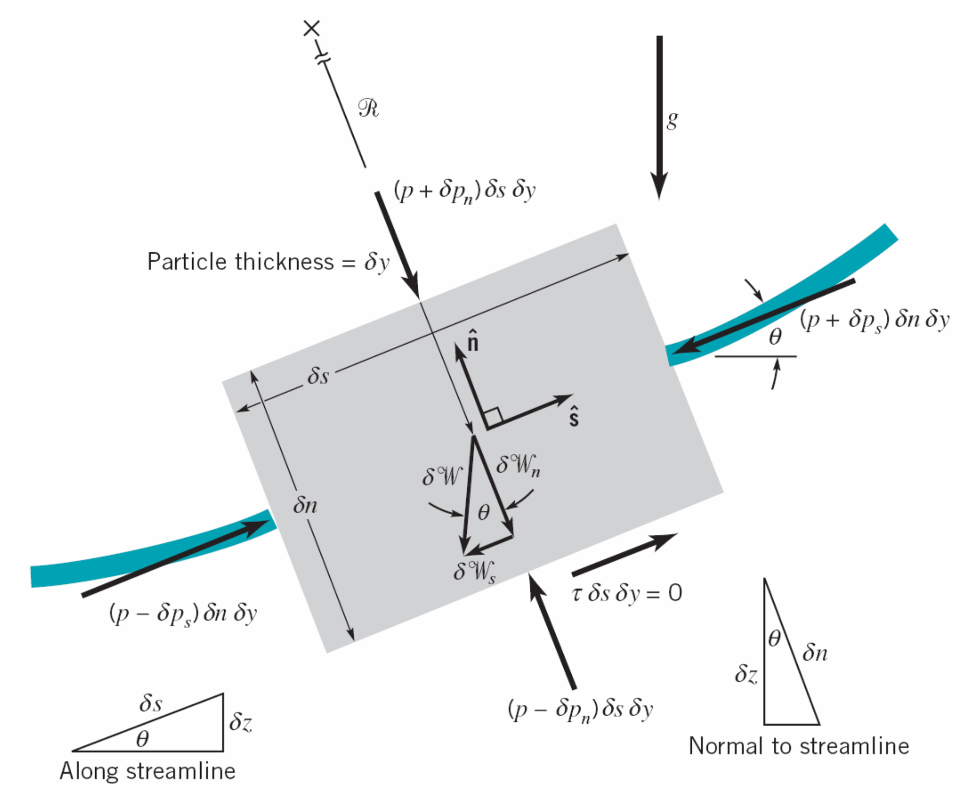


Acceleration along the streamline:

$$a_s = \frac{dv}{dt} = \frac{\partial v}{\partial s}\frac{\partial s}{\partial t} = \frac{\partial v}{\partial s}v$$

Centrifugal acceleration:

$$a_n = \frac{v^2}{R}$$



Along the streamline

$$\sum \delta F_s = m a_s = \rho \delta V \frac{\partial v}{\partial s} v$$

$$\sum \delta F_s = \delta W_s + \delta F_{ps} = -\rho g \delta V \sin(\theta) - \frac{\partial p}{\partial s} \delta V$$

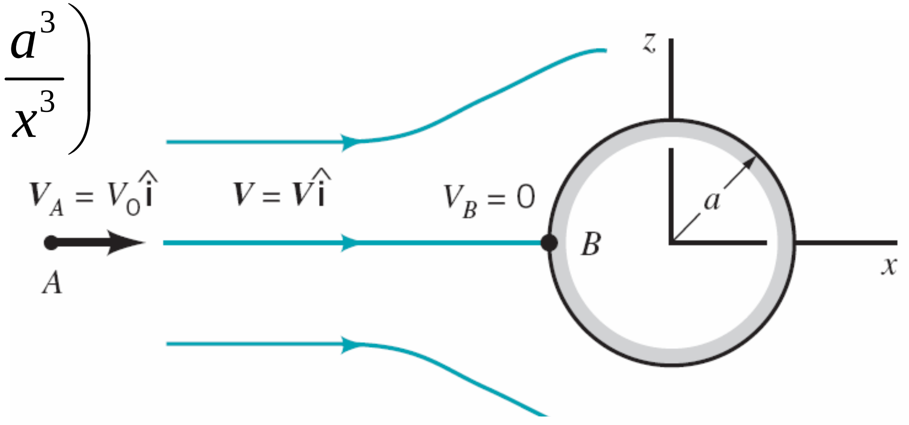
$$-\rho g \sin(\theta) - \frac{\partial p}{\partial s} = \rho \frac{\partial v}{\partial s} v$$

Balancing ball

Pressure variation along the streamline

- Consider inviscid, incompressible, steady flow along the horizontal streamline A-B in front of a sphere of radius a. Determine pressure variation along the streamline from point A to point B. Assume:

$$V = V_0\left(1+\frac{a^3}{x^3}\right)$$



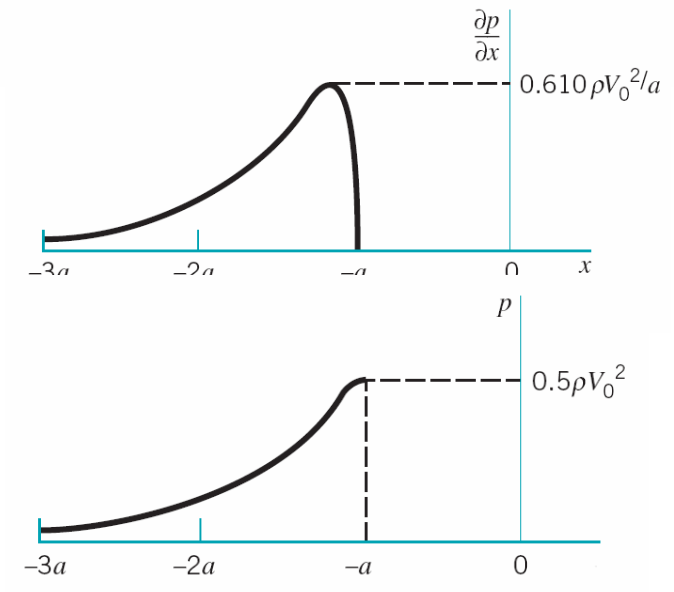
Equation of motion:

$$\frac{\partial p}{\partial s} = -\rho v \frac{\partial v}{\partial s}$$

$$v\frac{\partial v}{\partial s} = -3v_0^2\left(1+\frac{a^3}{x^3}\right)\frac{a^3}{x^4}$$

$$\frac{\partial p}{\partial x} = \frac{3\rho a^3 v_0^2}{x^4}\left(1+\frac{a^3}{x^3}\right)$$

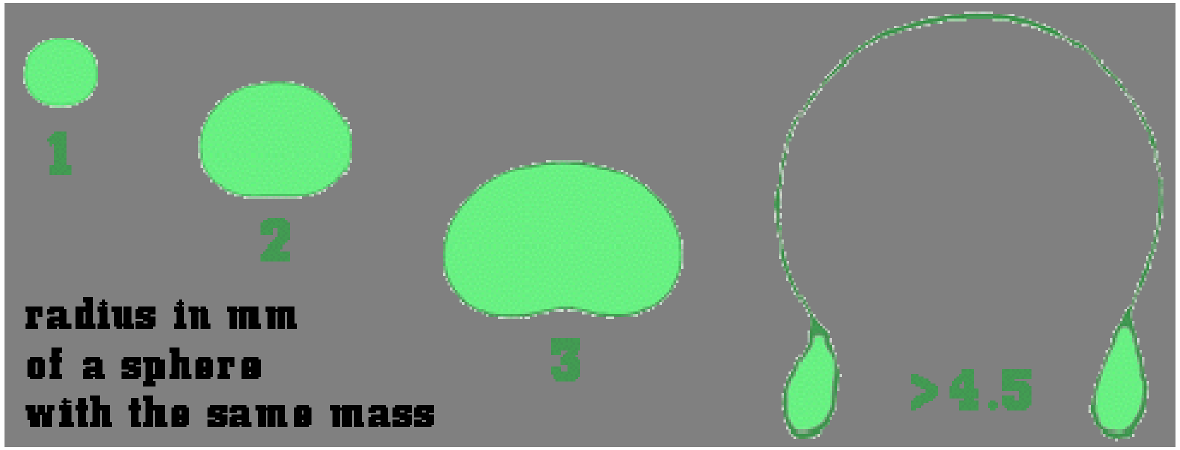
$$\Delta p = \int_{-\infty}^{-a}\frac{3\rho a^3 v_0^2}{x^4}\left(1+\frac{a^3}{x^3}\right)dx = -\rho v_0^2\left(\frac{a^3}{x^3}+\frac{1}{2}\left(\frac{a}{x}\right)^6\right)$$



(c)

Raindrop shape

The actual shape of a raindrop is a result of balance between the surface tension and the air pressure



Bernoulli equation

Integrating

$$-\rho g\sin(\theta)-\frac{\partial p}{\partial s}=\rho\frac{\partial v}{\partial s}v$$

$$\frac{dz}{ds}$$

$$\frac{dp}{ds}\text{, n=const along streamline}$$

$$-\rho g\frac{dz}{ds}-\frac{dp}{ds}=\frac{1}{2}\rho\frac{dv^2}{ds}$$

We find

$$dp+\frac{1}{2}\rho d(v^2)+\rho g dz=0$$

Along a streamline

Assuming incompressible flow:

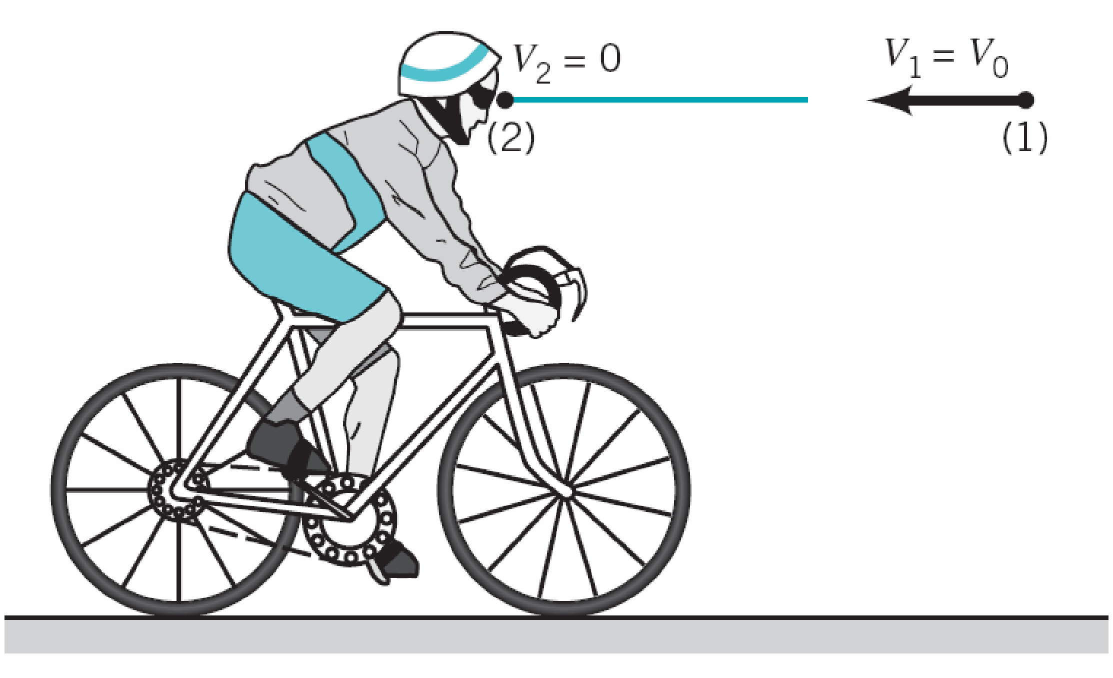
$$p+\frac{1}{2}\rho v^2+\rho g z=const$$

Along a streamline

Bernoulli equation

Example: Bicycle

- Let's consider coordinate system fixed to the bike.
  Now Bernoulli equation can be applied to



$$\boxed{p_2 - p_1 = \frac{1}{2}\rho v_0^{\ 2}}$$

Pressure variation normal to streamline

$$\sum \delta F_n = m a_n = \rho \delta V \frac{v^2}{R}$$

$$\sum \delta F_n = \delta W_n + \delta F_{pn} = -\rho g \delta V \cos(\theta) - \frac{\partial p}{\partial n} \delta V$$

$$-\rho g \frac{dz}{dn} - \frac{\partial p}{\partial n} = \rho \frac{v^2}{R}$$

$$p + \rho \int \frac{v^2}{R} dn + \rho g z = const$$

Across streamlines

compare

$$p + \frac{1}{2} \rho v^2 + \rho g z = const$$

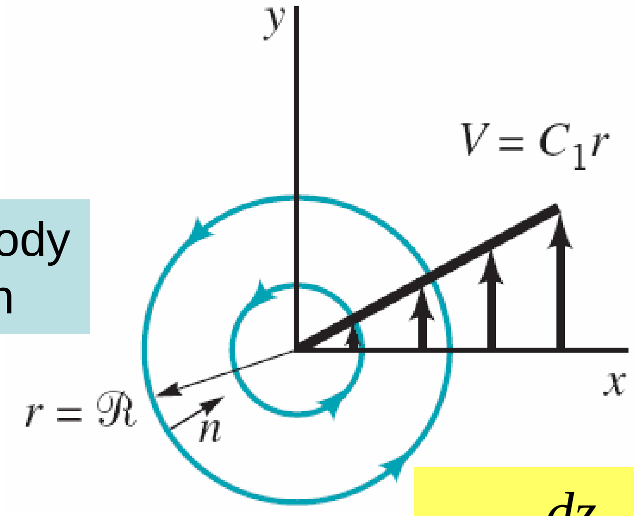
Along a streamline

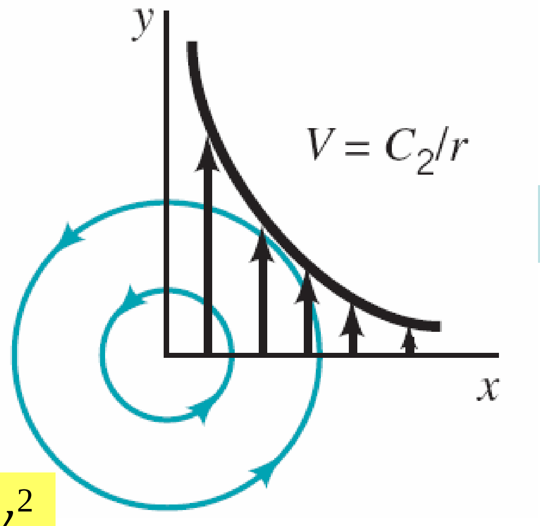
Free vortex

Example: pressure variation normal to streamline

- Let’s consider 2 types of vortices with the velocity distribution as below:

solid body rotation



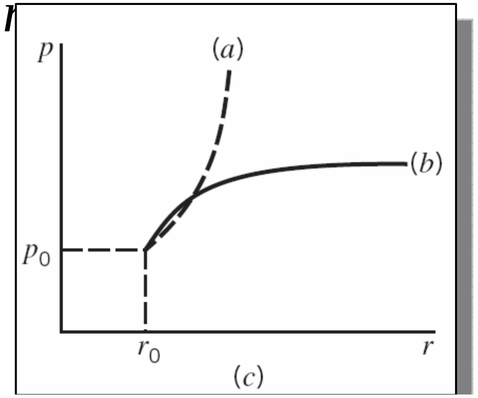


free vortex

$$-\rho g \frac{dz}{dn} - \frac{\partial p}{\partial n} = \rho \frac{v^2}{R}$$

as $\frac{\partial}{\partial n} = -\frac{\partial}{\partial r}$, $\frac{\partial p}{\partial r} = \frac{\rho V^2}{r} = \rho C_1^2 r$

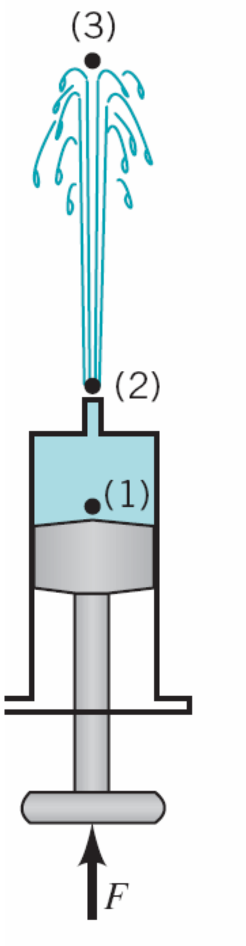
$$p = \frac{1}{2}\rho C_1^2 \left(r_0^2 - r^2\right) + p_0$$



$$\frac{\partial p}{\partial r} = \frac{\rho V^2}{r} = \rho \frac{C_1^2}{r^3}$$

$$p = \frac{1}{2}\rho C_2^2 \left(\frac{1}{r_0^2} - \frac{1}{r^2}\right) + p_0$$

$$\boxed{p + \frac{1}{2}\rho v^2 + \rho g z = const}$$



	Energy Type		
Point	**Kinetic** $\rho V^2/2$	**Potential** γz	**Pressure** p
1	Small	Zero	Large
2	Large	Small	Zero
3	Zero	Large	Zero

Static, Stagnation, Dynamic and TotalPressure

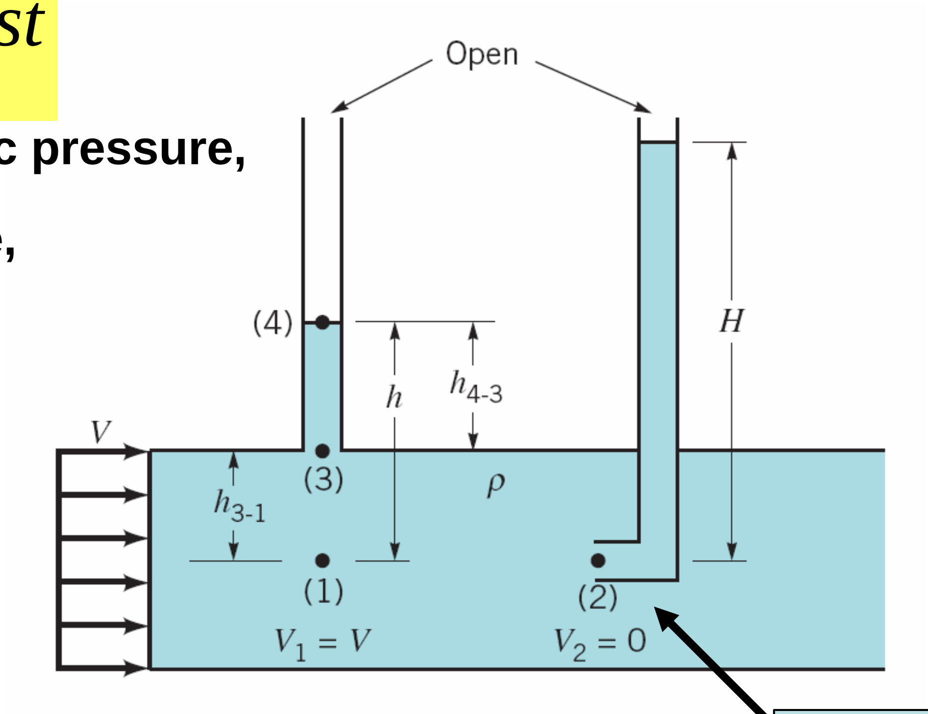
- each term in Bernoulli equation has dimensions of pressure and can be interpreted as some sort of pressure

$$p + \frac{1}{2}\rho v^2 + \rho g z = const$$

hydrostatic pressure,

dynamic pressure,

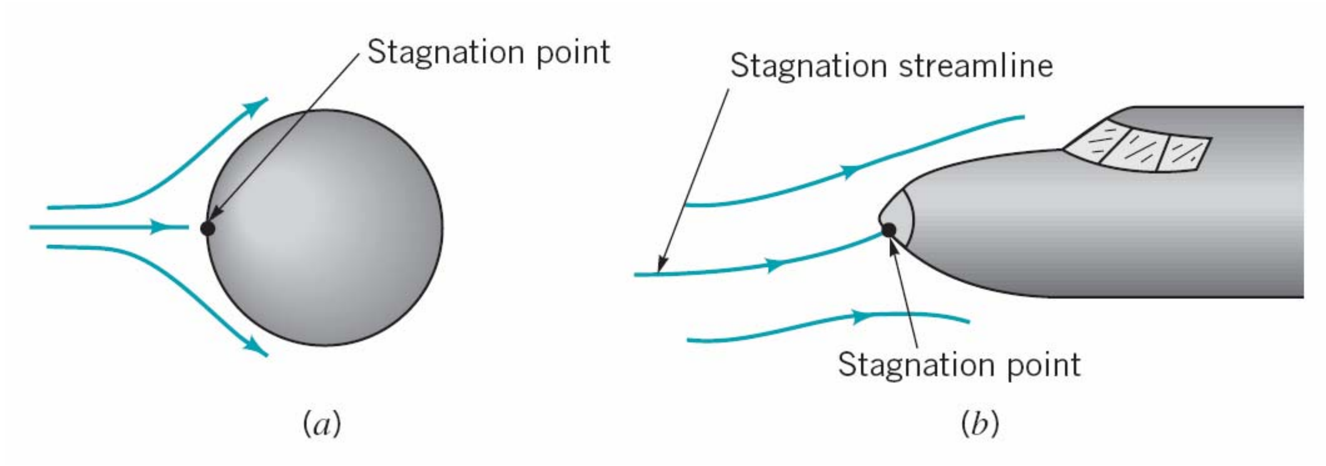
static pressure,
point (3)

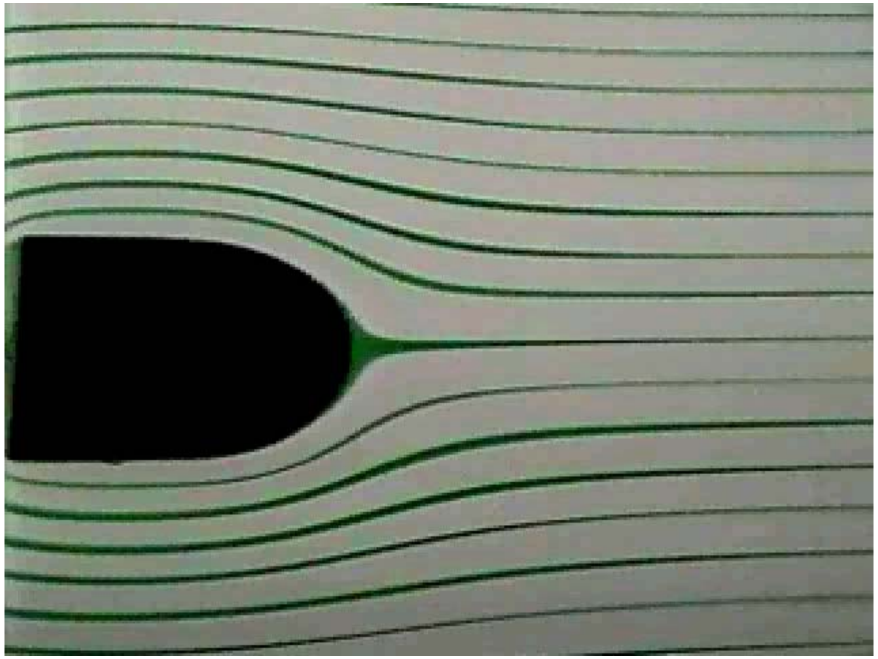


Velocity can be determined from stagnation pressure: ⇨

$$p_2 = p_1 + \frac{1}{2}\rho v^2$$

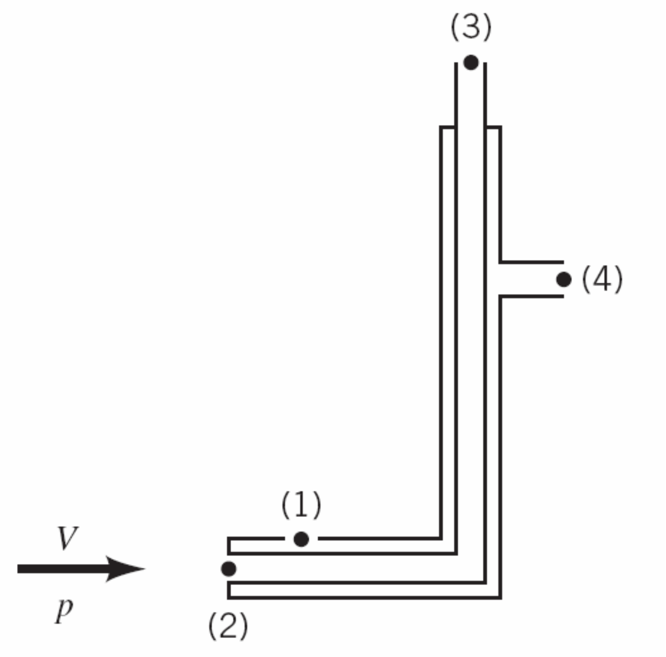
Stagnation pressure





On any body in a flowing fluid there is a stagnation point. Some of the fluid flows "over" and some "under" the body. The dividing line (the stagnation streamline) terminates at the stagnation point on the body.
As indicated by the dye filaments in the water flowing past a streamlined object, the velocity decreases as the fluid approaches the stagnation point. The pressure at the stagnation point (the stagnation pressure) is that pressure obtained when a flowing fluid is decelerated to zero speed by a frictionless process

Pitot-static tube



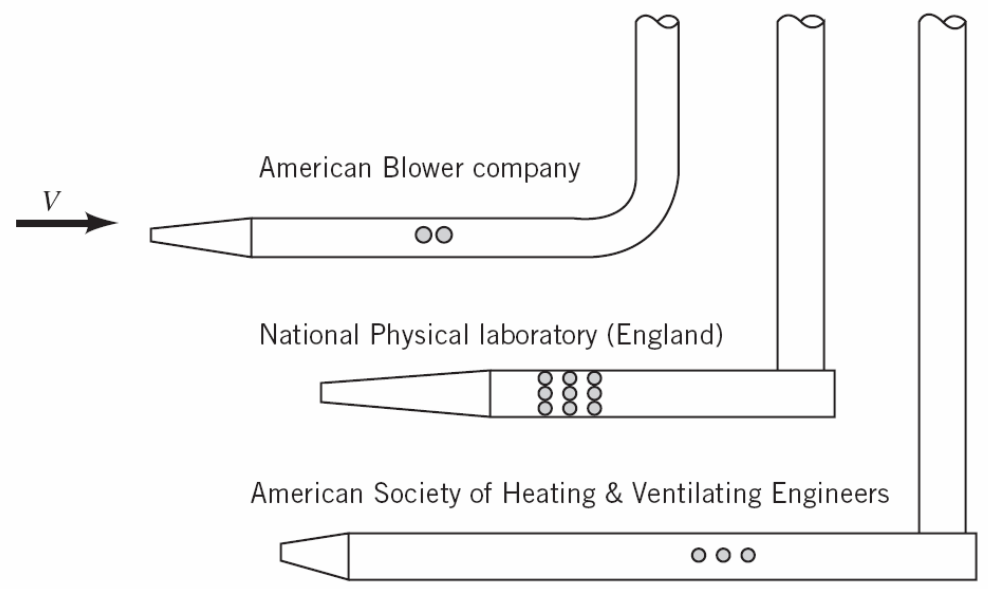
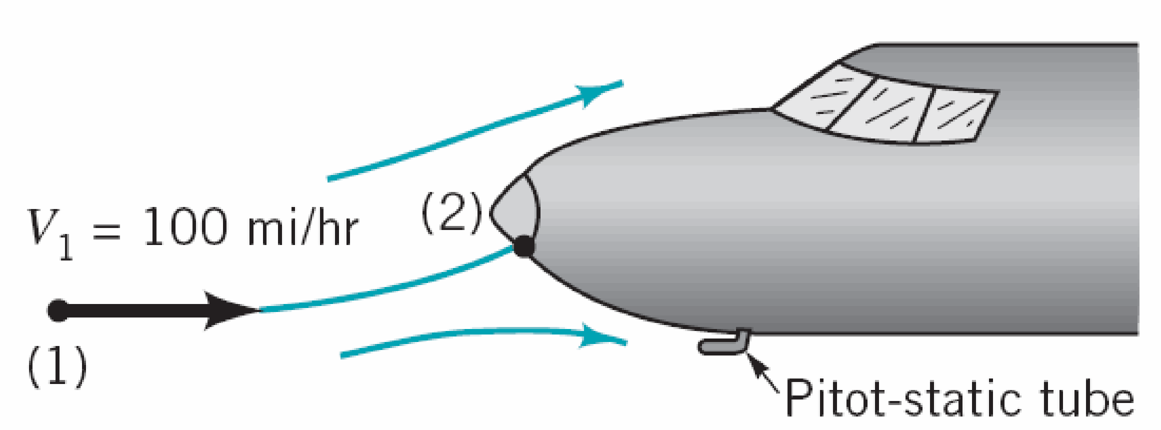
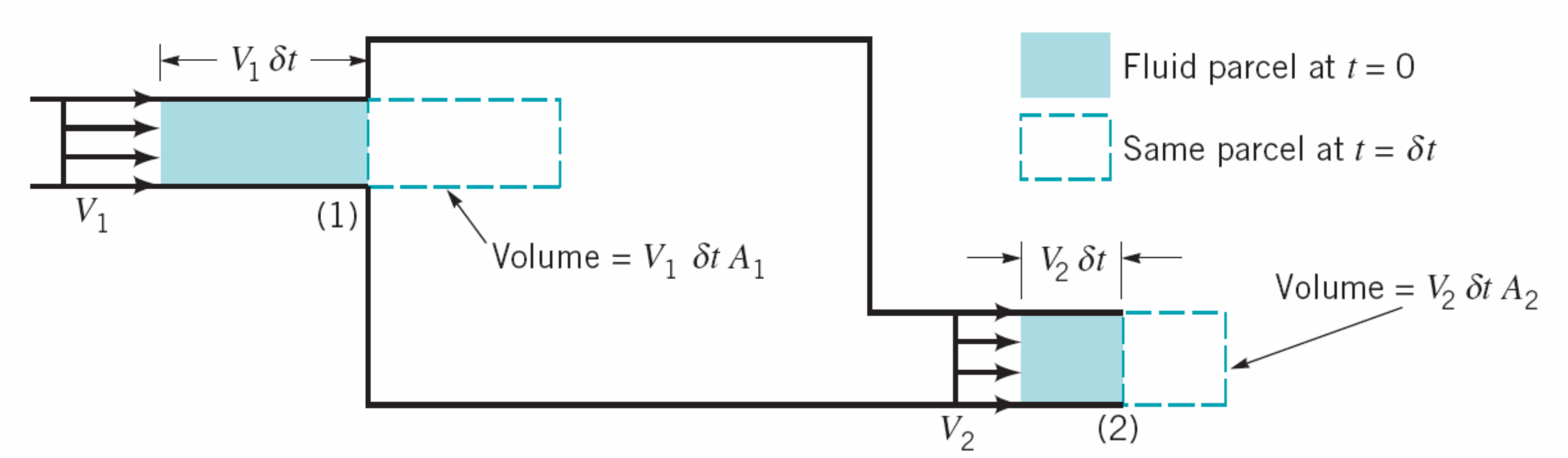


FIGURE 3.7 Typical Pitot-static tube designs.

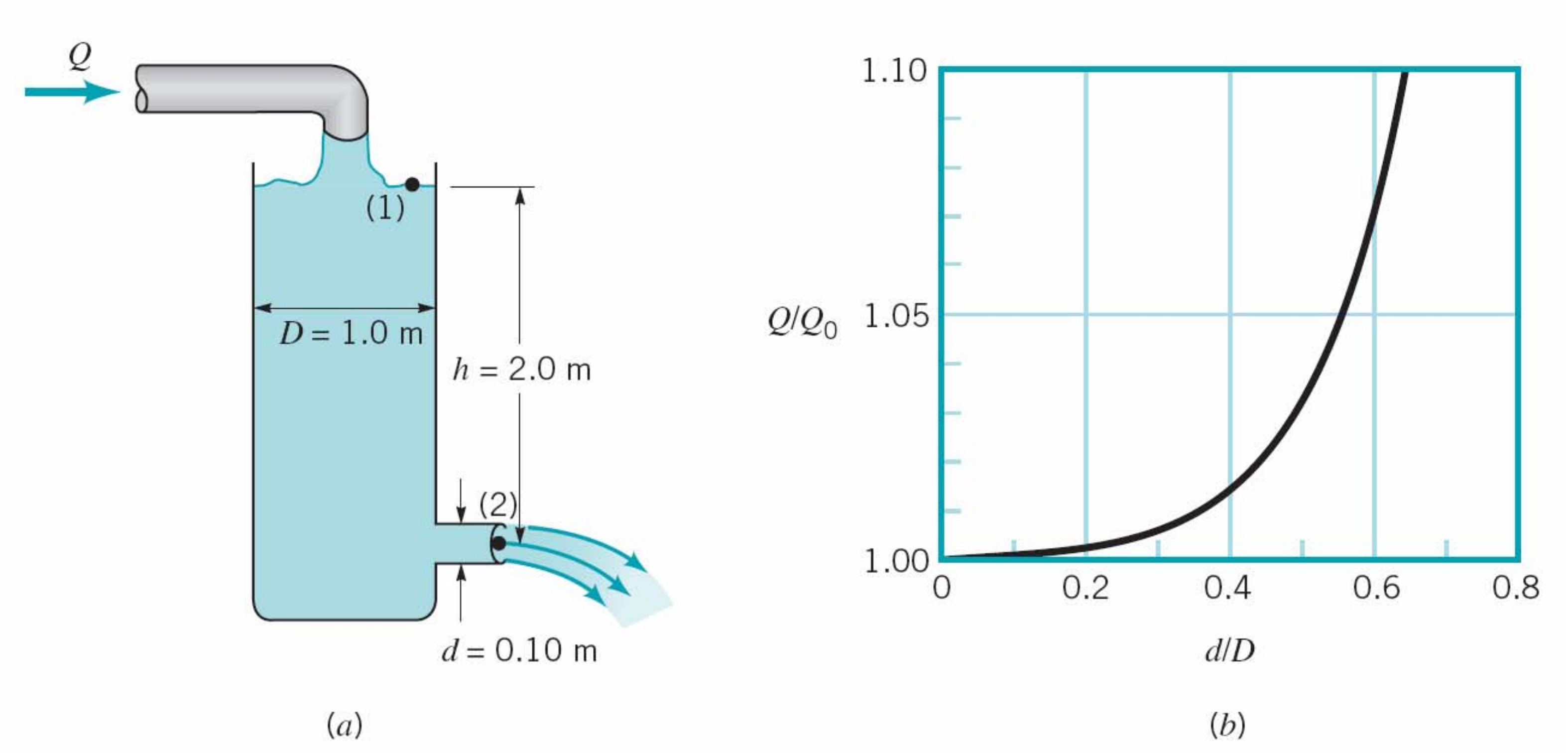


Steady flow into and out of a tank.



$$\boxed{\rho_1 A_1 v_1 = \rho_2 A_2 v_2}$$

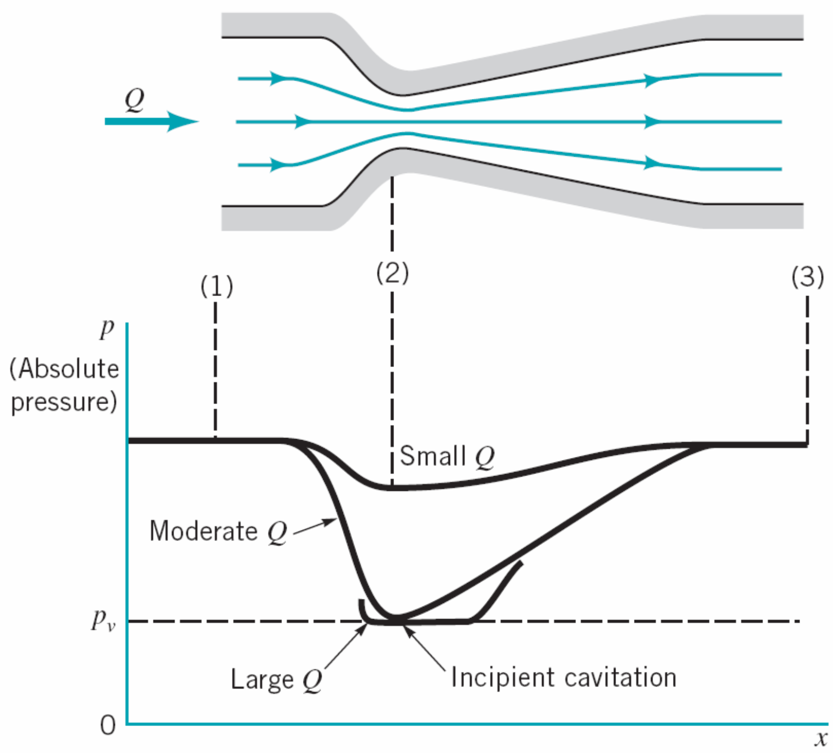
Determine the flow rate to keep the height constant

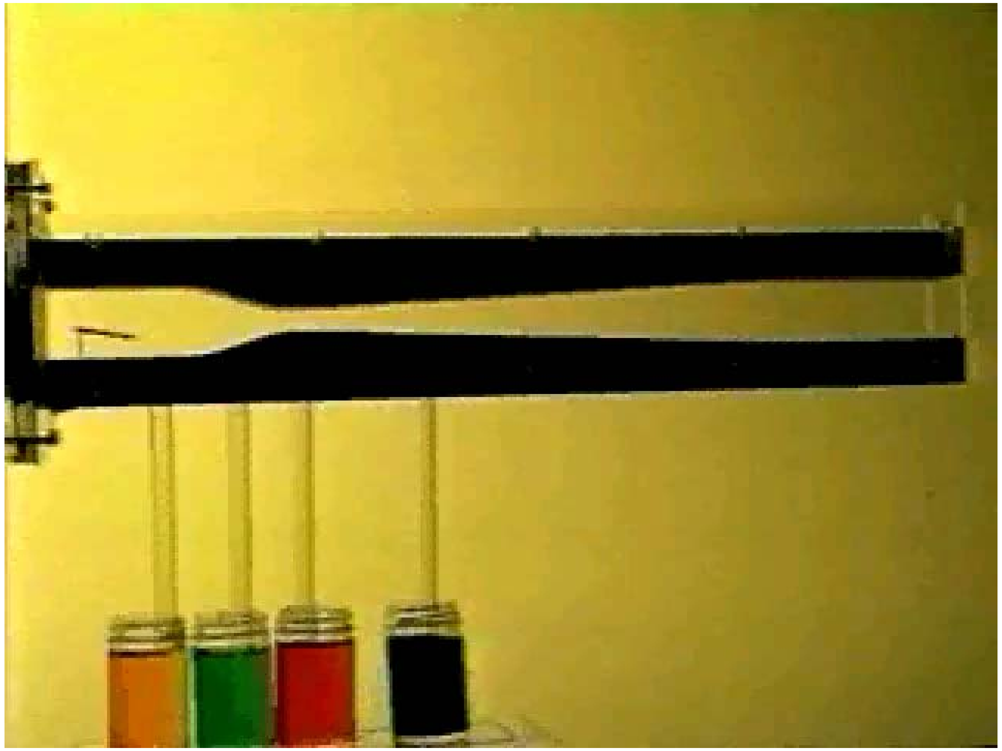


$$p_1 + \frac{1}{2}\rho v_1{}^2 + \rho g z_1 = p_2 + \frac{1}{2}\rho v_2{}^2 + \rho g z_2$$

$$Q = A_1 v_1 = A_2 v_2$$

Venturi channel

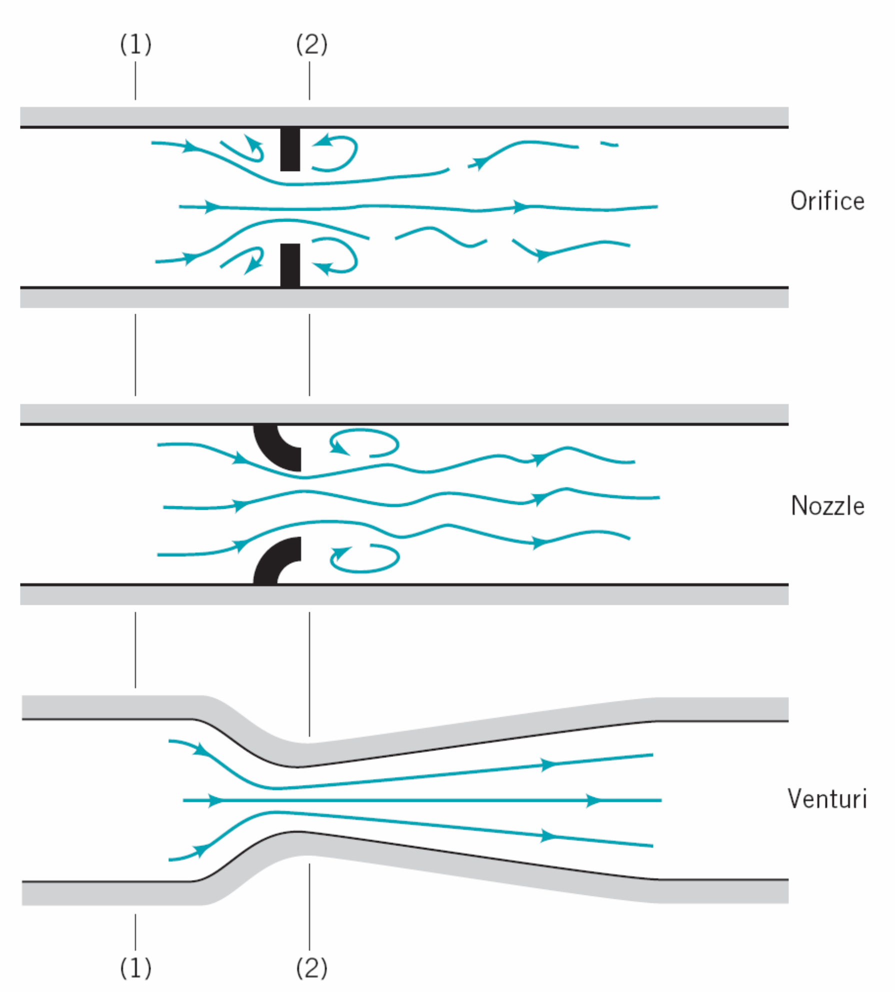




Measuring flow rate in pipes

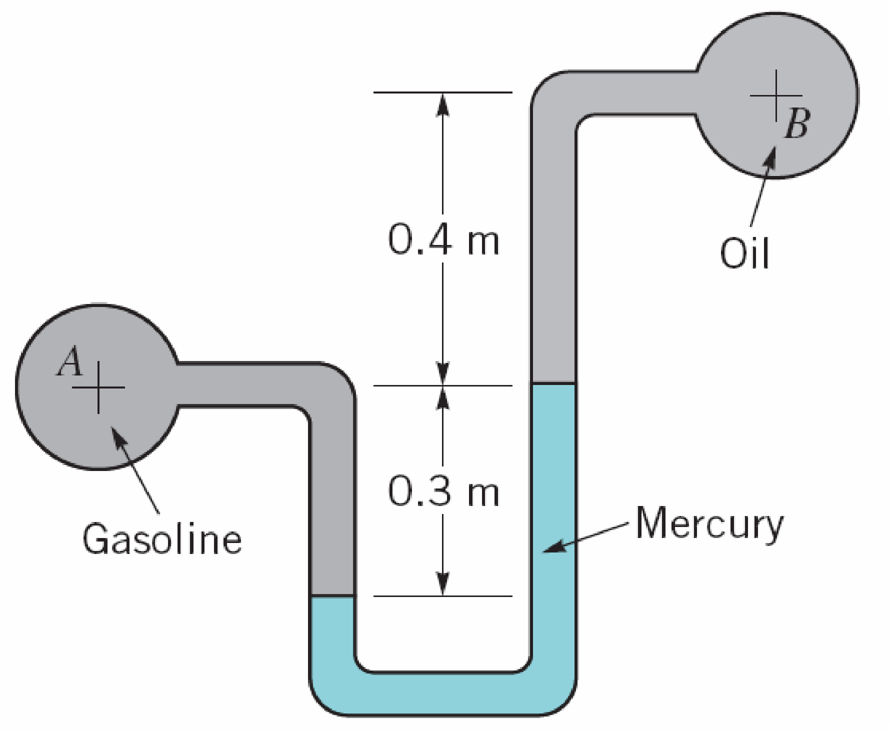
$$p_1 + \frac{1}{2}\rho v_1{}^2 = p_2 + \frac{1}{2}\rho v_2{}^2$$

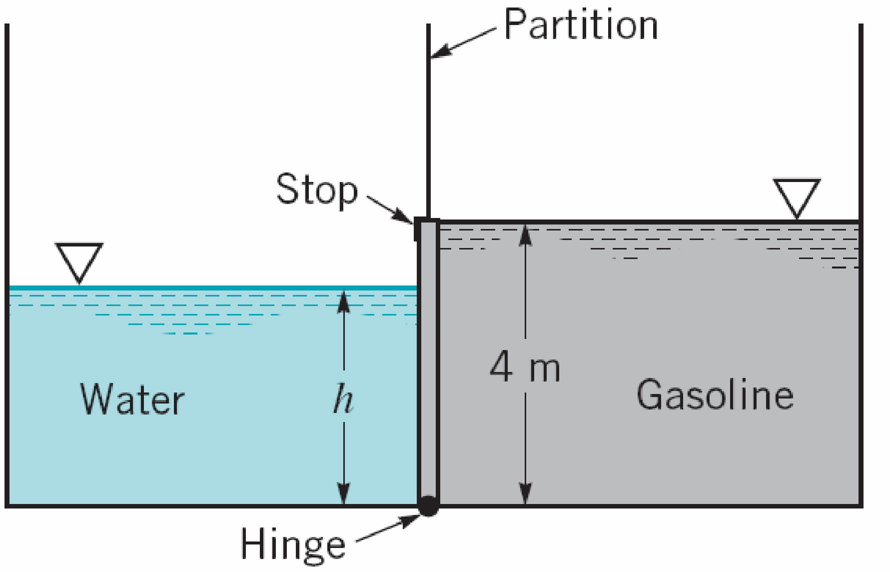
$$Q = A_1 v_1 = A_2 v_2$$



Probelms

- 2.43 Pipe A contains gasoline (SG=0.7), pipe B contains oil (SG=0.9). Determine new differential reading of pressure in A decreased by 25 kPa.
- 2.61 An open tank contains gasoline ρ=700kg/cm at a depth of 4m. The gate is 4m high and 2m wide. Water is slowly added to the empty side of the tank. At what depth *h* the gate will open.





Problems

- **3.71** calculate *h*. Assume water inviscid and incompressible.