Lecture 3

Fluid Kinematics: Velocity field, Acceleration, Reynolds Transport Theorem and its application

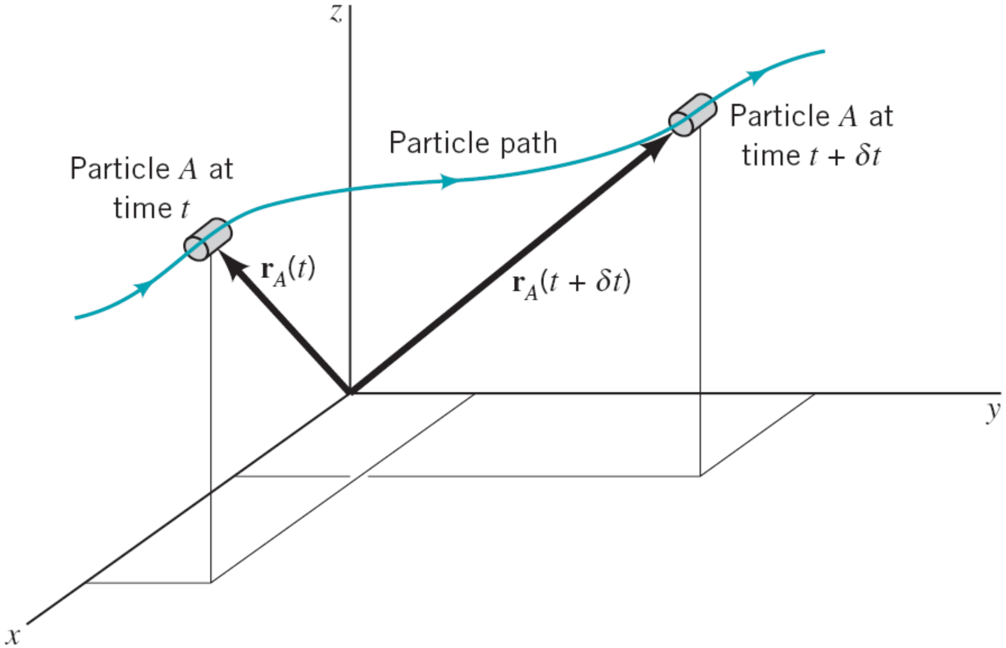
Aims

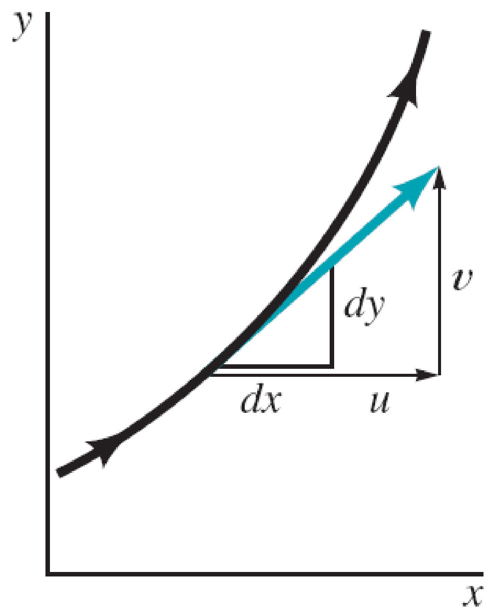
- Describing fluid flow as a field
- How the flowing fluid interacts with the environment (forces and energy)

Lecture plan

- Describing flow with the fields: Eulerian vs. Lagrangian description.
- Flow analysis: Streamlines, Streaklines, Pathlines.
- How to perform calculations in the field description: the Material Derivative
- Reynold’s Transport Theorem
- Application of Reynolds transport theorem: Continuity, Momentum and Energy conservation

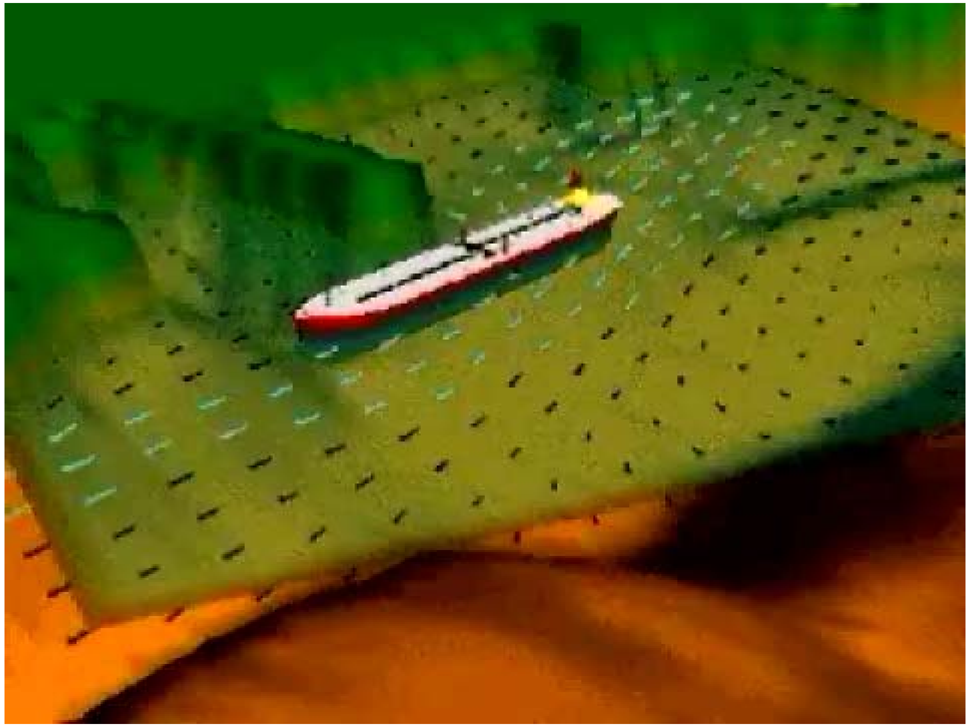
Velocity field





$$\frac{dy}{dx} = \frac{v}{u}$$

Velocity field: example



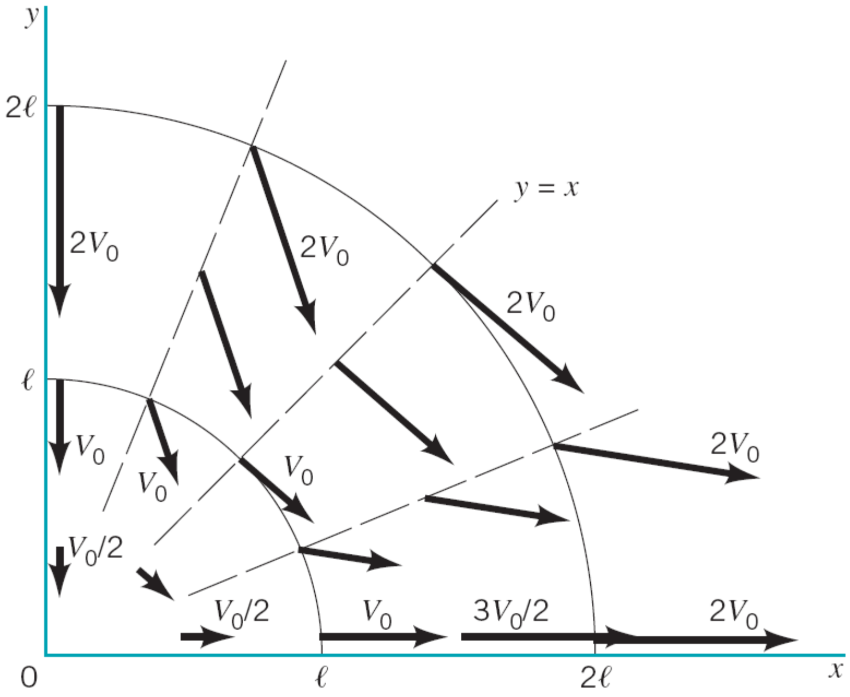
The calculated velocity field in a shipping channel is shown as the tide comes in and goes out. The fluid speed is given by the length and color of the arrows. The instantaneous flow direction is indicated by the direction that the velocity arrows point.

Velocity field representation

Velocity field is given by:

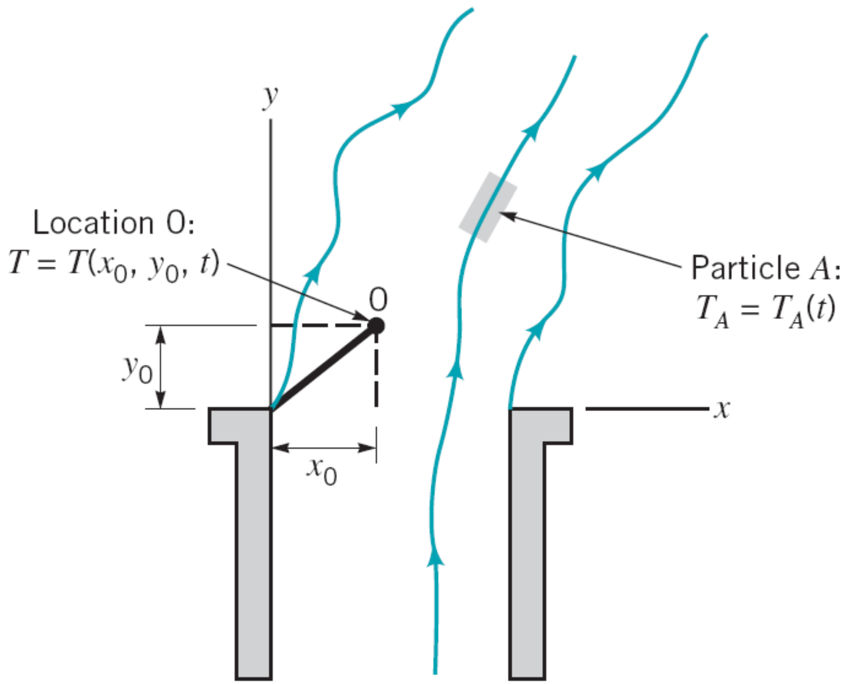
$$\vec{v} = (v_0 / l)(x\vec{i} - y\vec{j})$$

- Sketch the field in the first quadrant
- find where velocity will be equal to $\boldsymbol{v_0}$



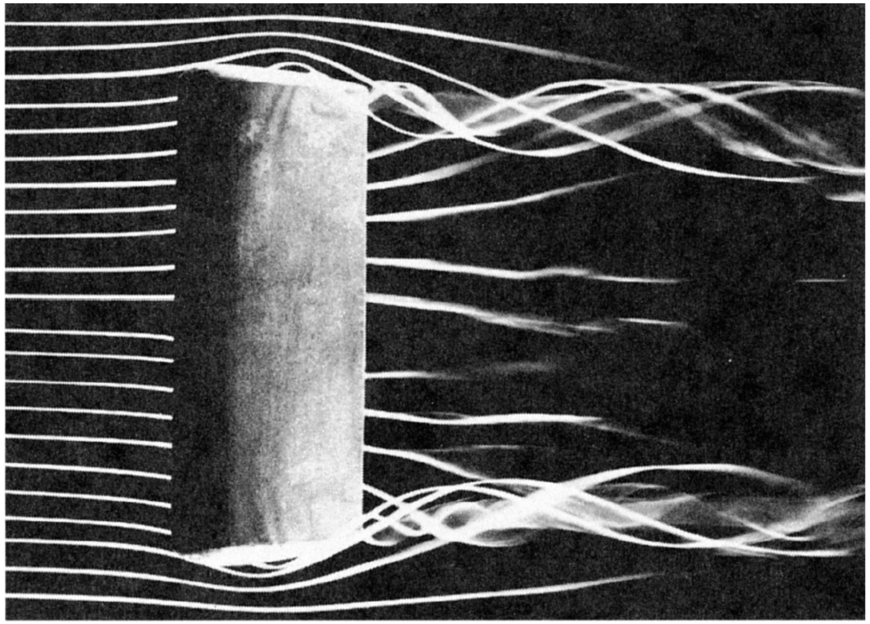
Eulerian and Lagrangian flow description

- Eulerian method – field concept is used, flow parameters (T, P, v etc.) are measured in every point in space vs. time
- Lagrangian method – an individual fluid particle is followed, parameters associated with this particle are followed in time



Example: Smoke coming out of a chimney

1D, 2D and 3D flow

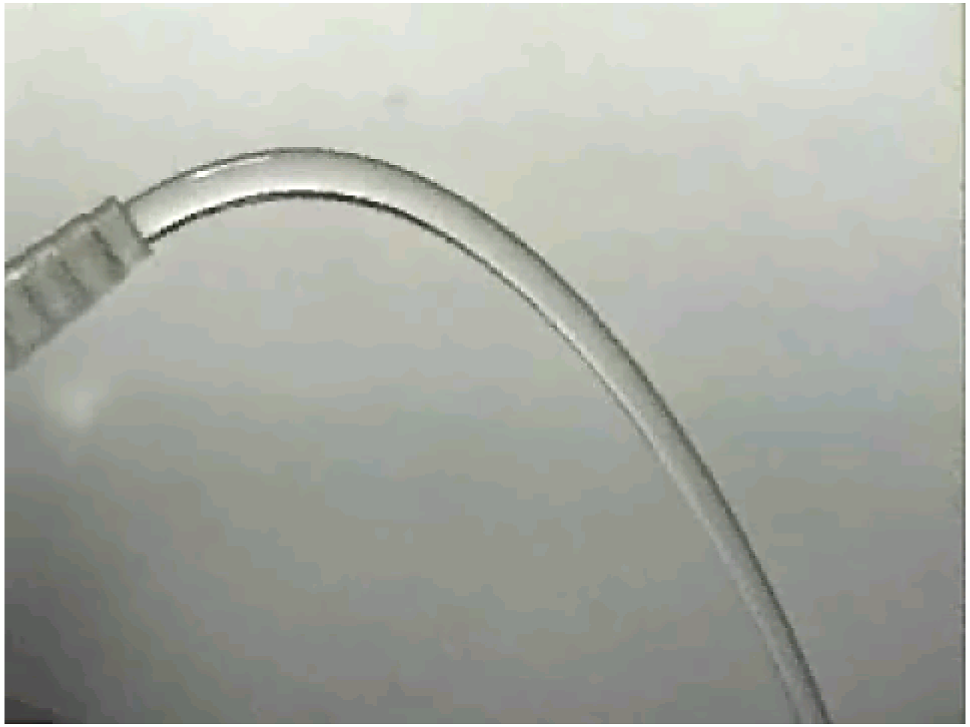


Flow visualization of the complex three-dimensional flow past a model airfoil

The flow generated by an airplane is made visible by flying a model Airbus airplane through two plumes of smoke. The complex, unsteady, three-dimensional swirling motion generated at the wing tips (called trailing vorticies) is clearly visible

Flow types

- **Steady flow** – the velocity at any given point in space doesn't vary with time. Otherwise the flow is called **unsteady**
- **Laminar flow** – fluid particles follow well defined pathlines at any moment in time, in **turbulent flow** pathlines are not defined.



Streamlines

Velocity field is given by:

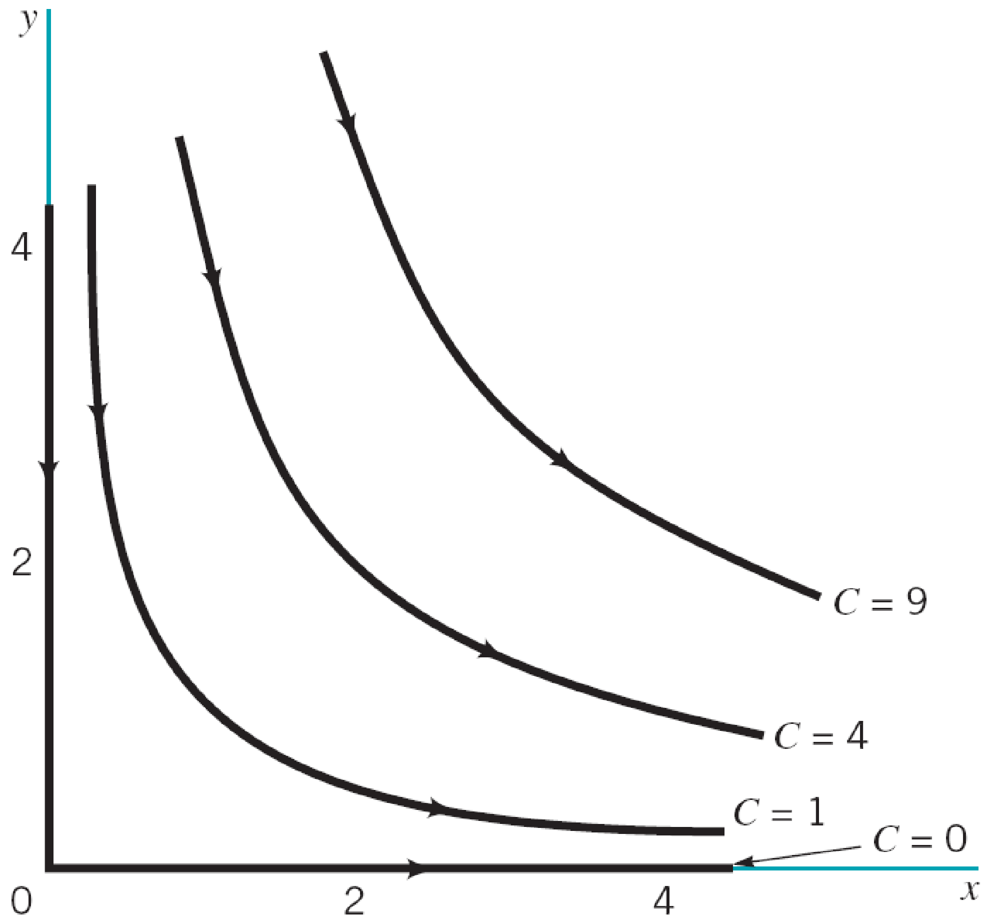
$$\vec{v} = (v_0 / l)(x\vec{i} - y\vec{j})$$

•draw the streamlines and find there equation

$$\frac{dy}{dx} = \frac{v}{u}$$

$$\frac{dy}{dx} = \frac{v}{u} = -\frac{y}{x};$$

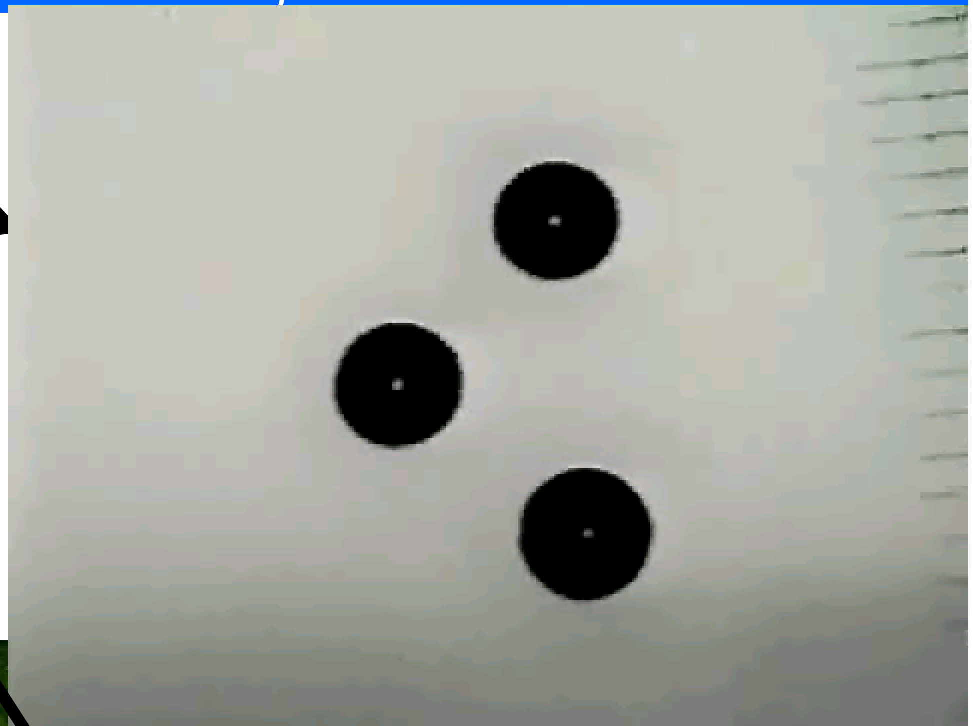
$$\ln y = \ln x + C;\ y = C' / x$$



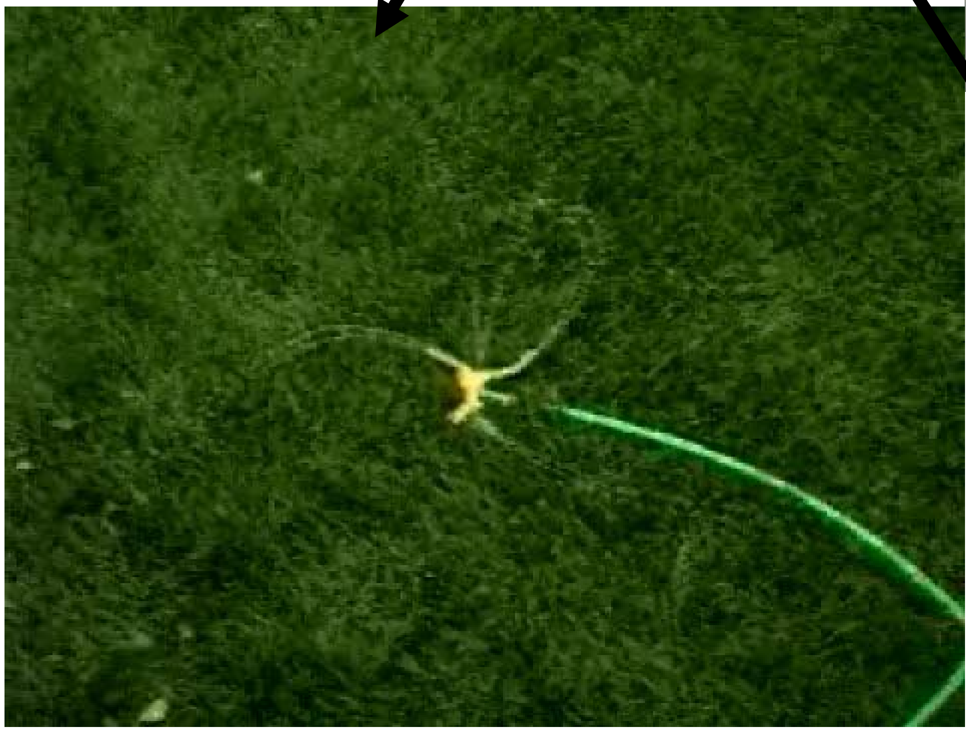
$$xy = Const$$

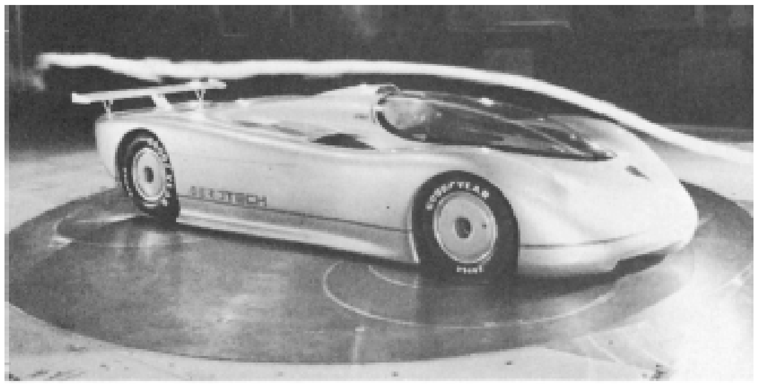
Streamlines, Streaklines, Pathlines

- Streamline – line that everywhere tangent to velocity field
- Streakline – all particles that passed through a common point
- Pathline – line traced by a given particle as it flows



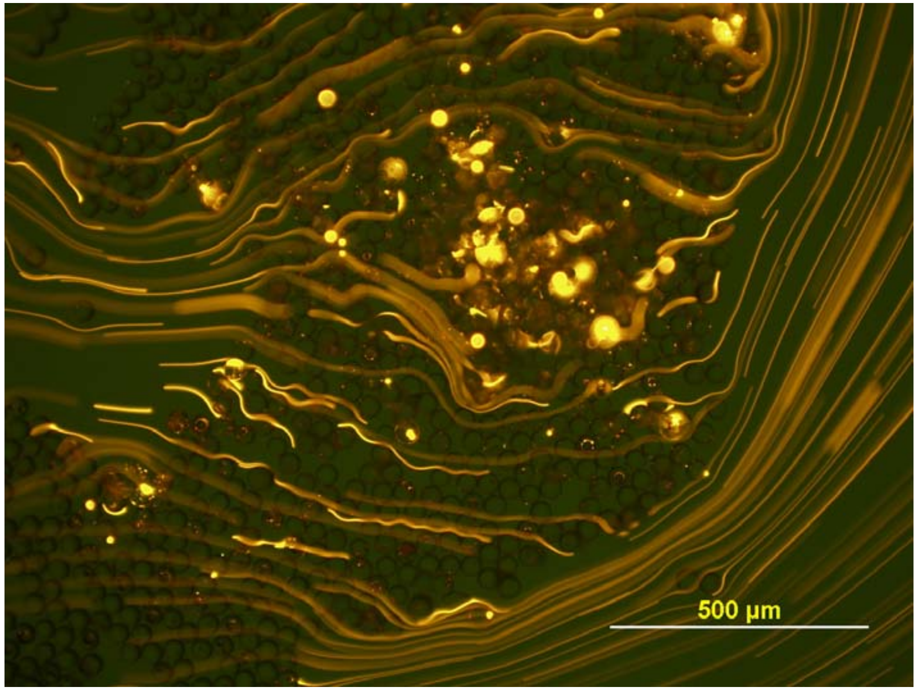
streamlines



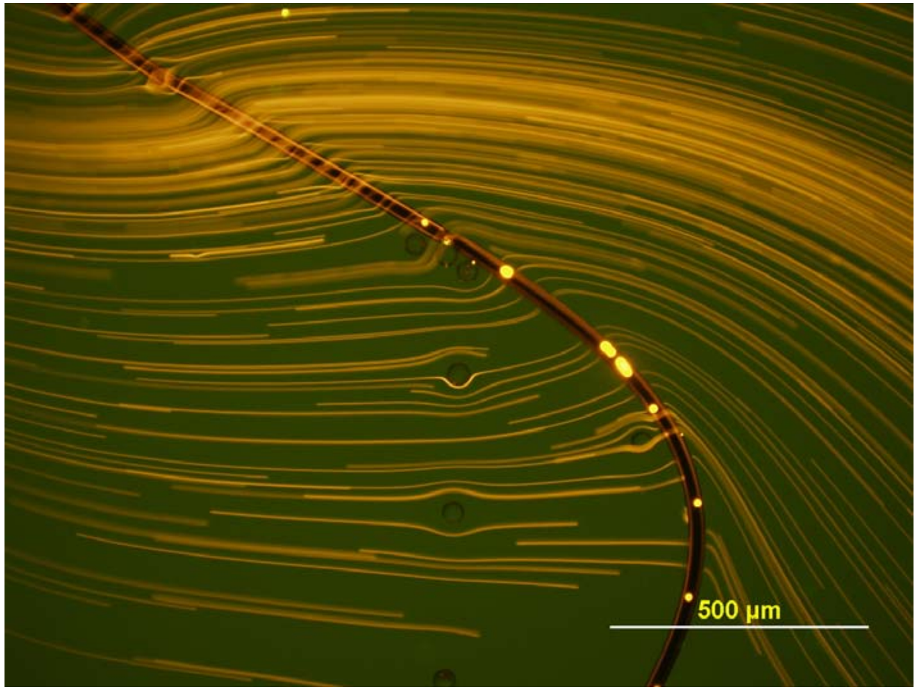


Imaging flow in a microfluidic channel

- Flow is seeded with fluorescent particles and imaged… (Project 5th semester Fall 2006)



Flow through a loosely packed microspheres bed

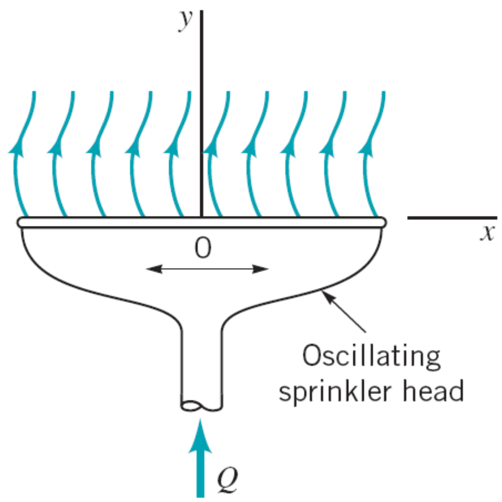


Flow at a channel turn.
Flow is disturbed by a microwire

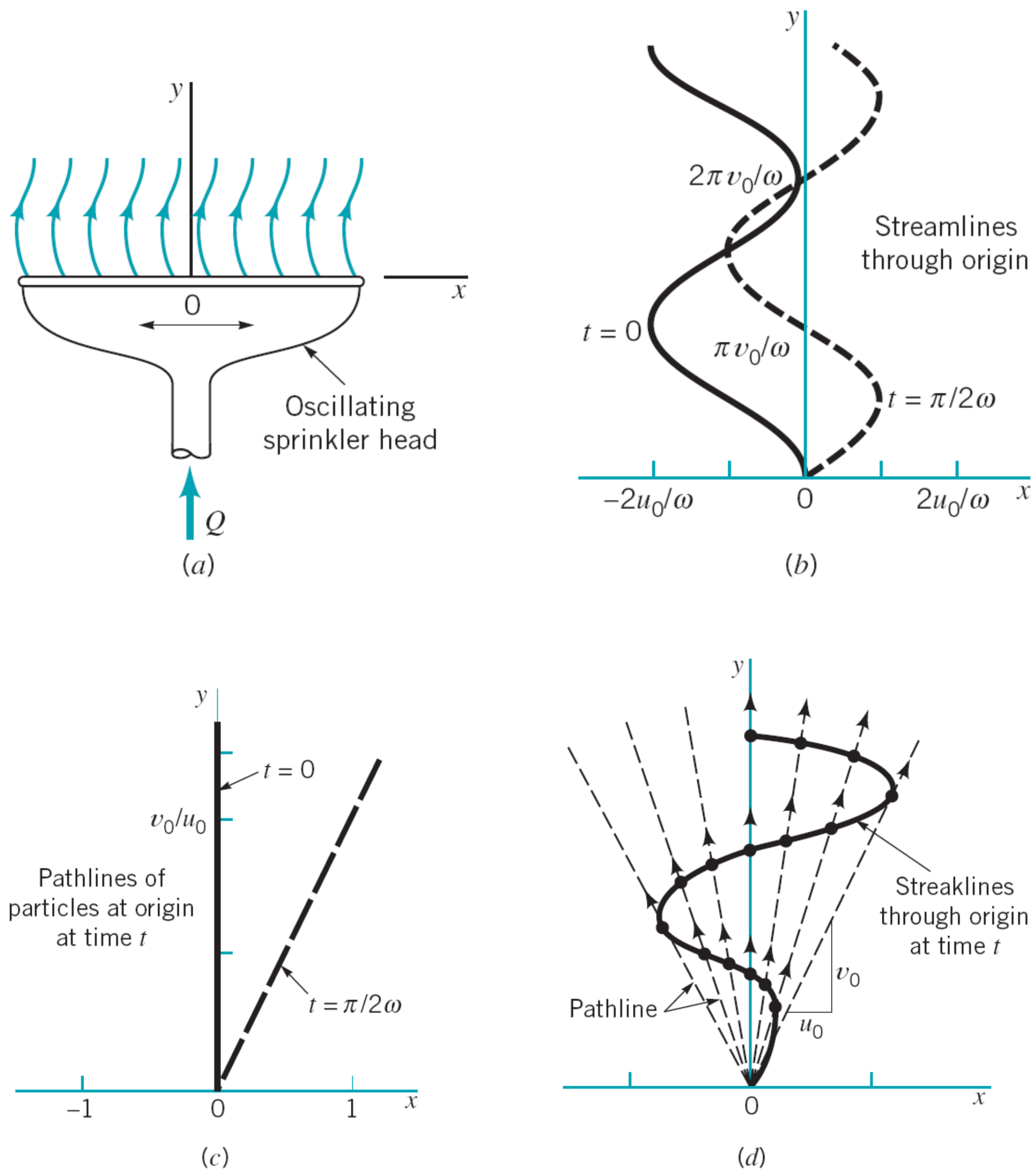
Example

Water flowing from an oscillating slit:

$$\vec{v} = u_0 \sin(\omega(t - y / v_0))\vec{i} + v_0 \vec{j}$$



(a)

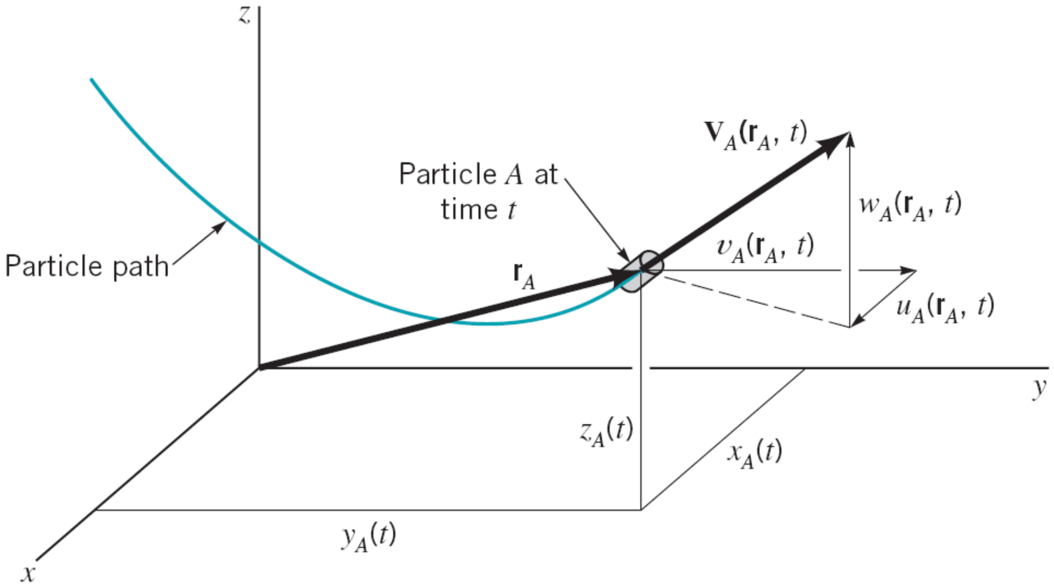


y
x
0
Oscillating sprinkler head
Q
(a)
y
2π v0/ω
Streamlines through origin
t = 0
π v0/ω
t = π/2ω
−2u0/ω
0
2u0/ω
x
(b)
y
t = 0
v0/u0
Pathlines of particles at origin at time t
t = π/2ω
−1
0
1
x
(c)
y
Streaklines through origin at time t
v0
u0
Pathline
0
x
(d)

Material derivative

$$V_A(r_A,t)$$

- Particle velocity



- Particle acceleration

$$a_A(r_A,t) = \frac{dV_A}{dt} = \frac{\partial V_A}{\partial t} + \frac{\partial V_A}{\partial x}\frac{\partial x_A}{\partial t} + \frac{\partial V_A}{\partial y}\frac{\partial y_A}{\partial t} + \frac{\partial V_A}{\partial z}\frac{\partial z_A}{\partial t}$$

$$\mathbf{a}(r,t) = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = \frac{D\mathbf{V}}{Dt}$$

- Material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + (\mathbf{V} \bullet \nabla)$$

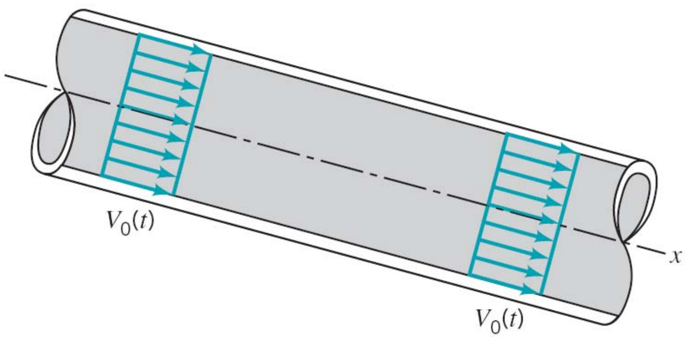
Material derivative

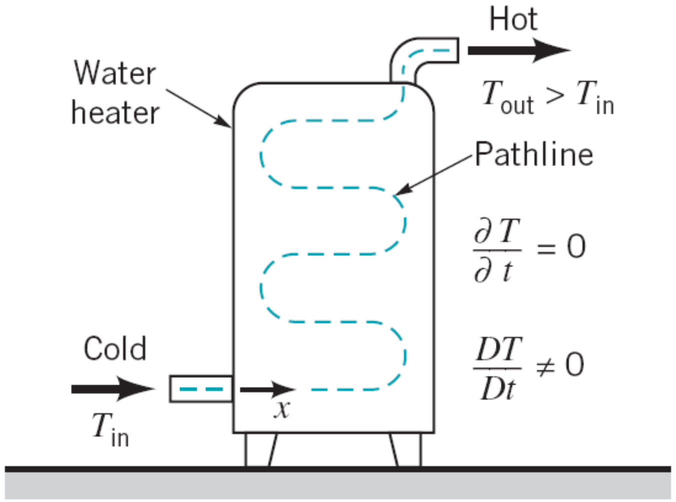
- is the rate of changes for a given variable with time for a given particle of fluid.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + (\mathbf{V} \bullet \nabla)$$

Unsteadiness of the flow (local acceleration)

Convective effect (convective acceleration)





Example: acceleration

Velocity field is given by: $\vec{v} = (v_0 / l)(x\vec{i} - y\vec{j})$

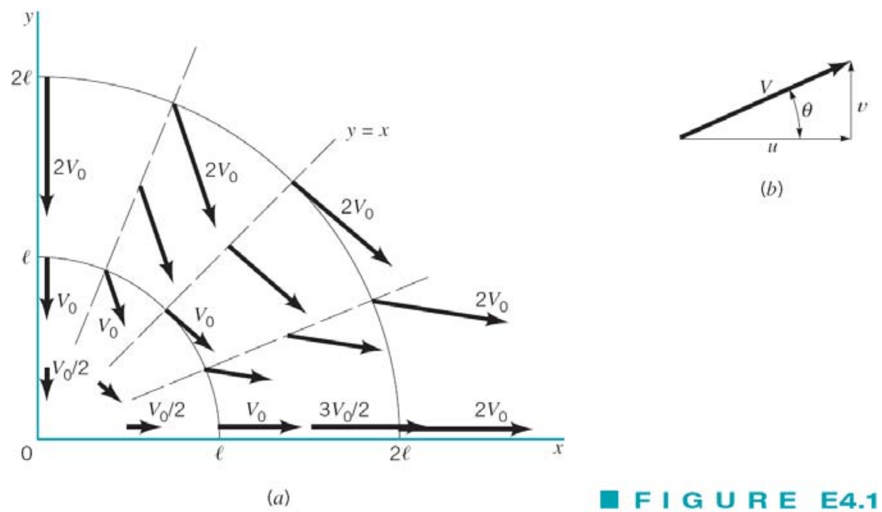
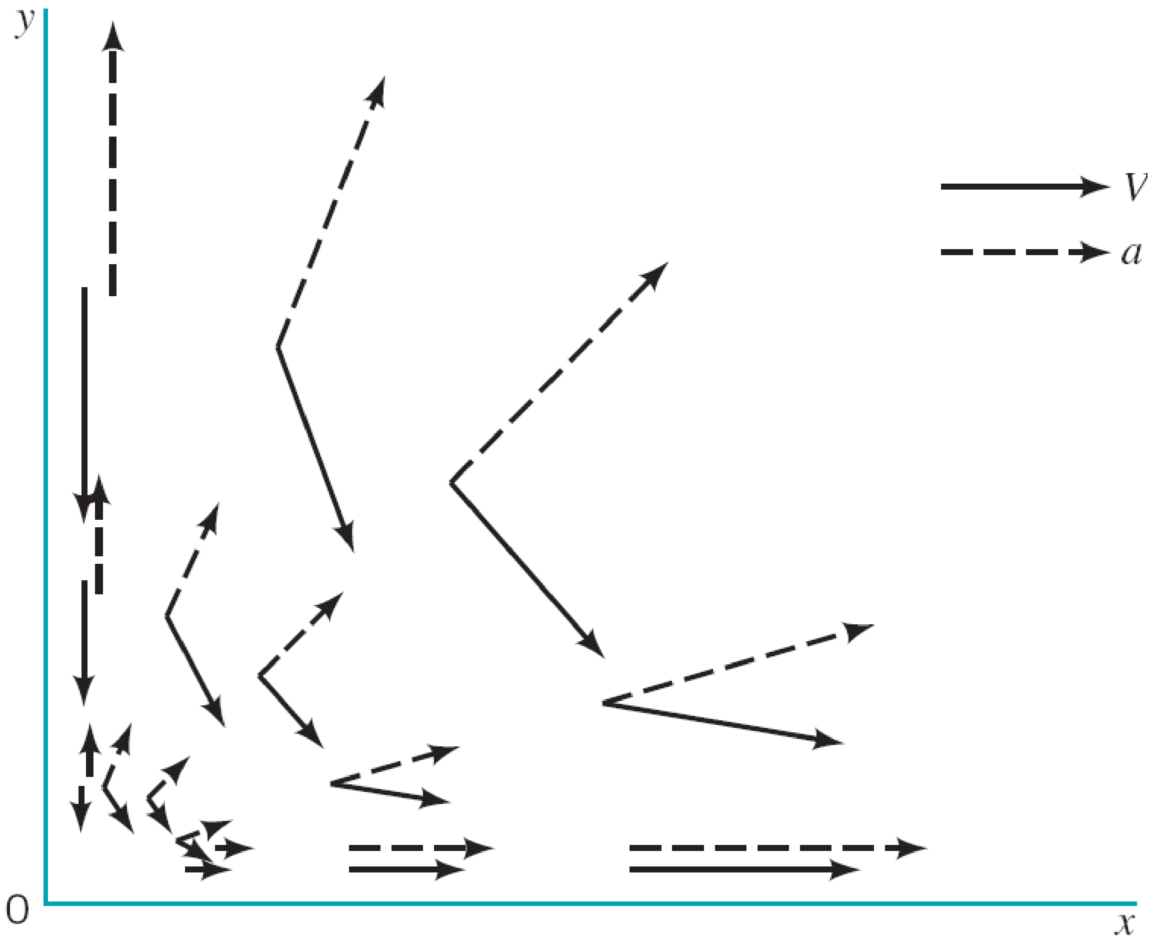


FIGURE E4.1

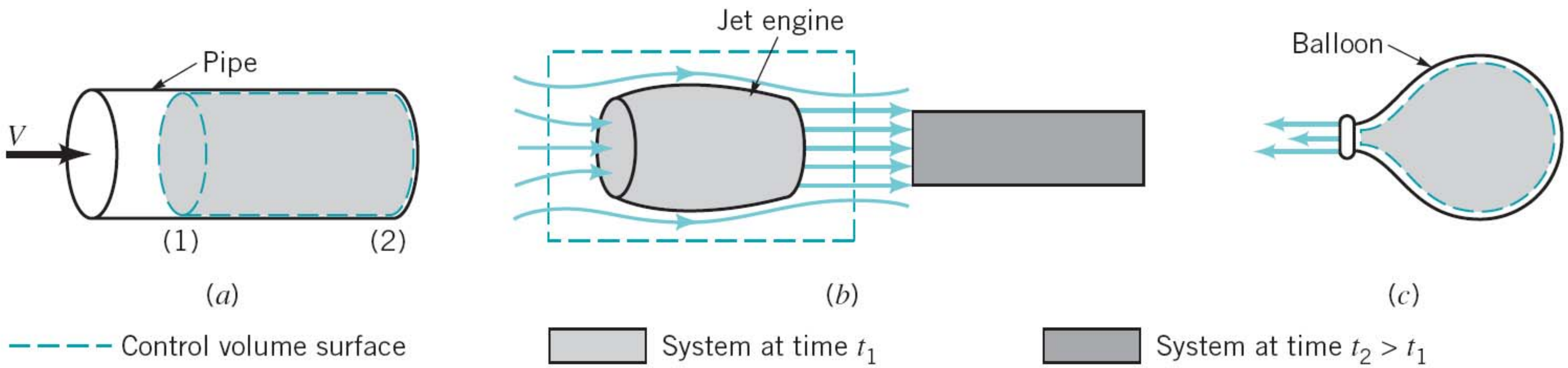
Find the acceleration and draw it on scheme



y
V
a
0
x

Control volume and system representation

- System – specific identifiable quantity of matter, that might interact with the surrounding but always contains the same mass
- Control volume – geometrical entity, a volume in space through which fluid may flow



Governing laws of fluid motion are stated in terms in system, but control volume approach is essential for practical applications

Control volume: example

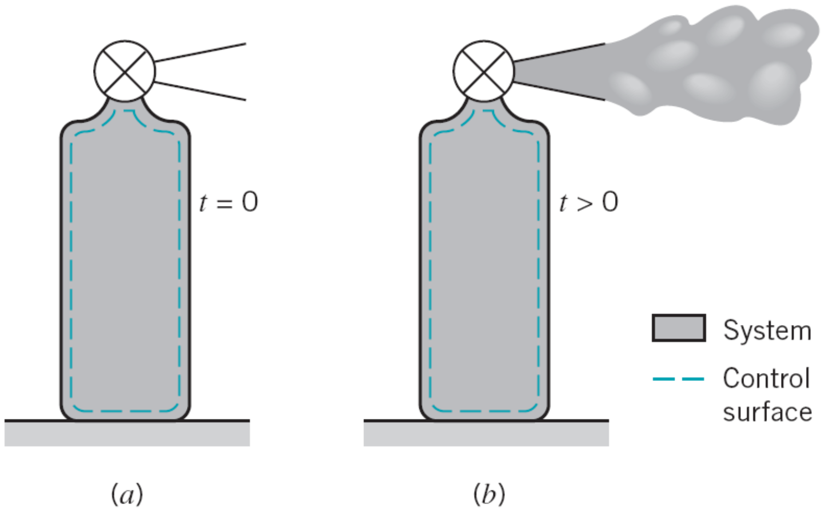
- Extensive property: B=mb (e.g. m (b=1), mv (b=v), mv^2/2 etc)

intensive property

$$B_{sys} = \int_{sys} \rho b d\mathcal{V}$$

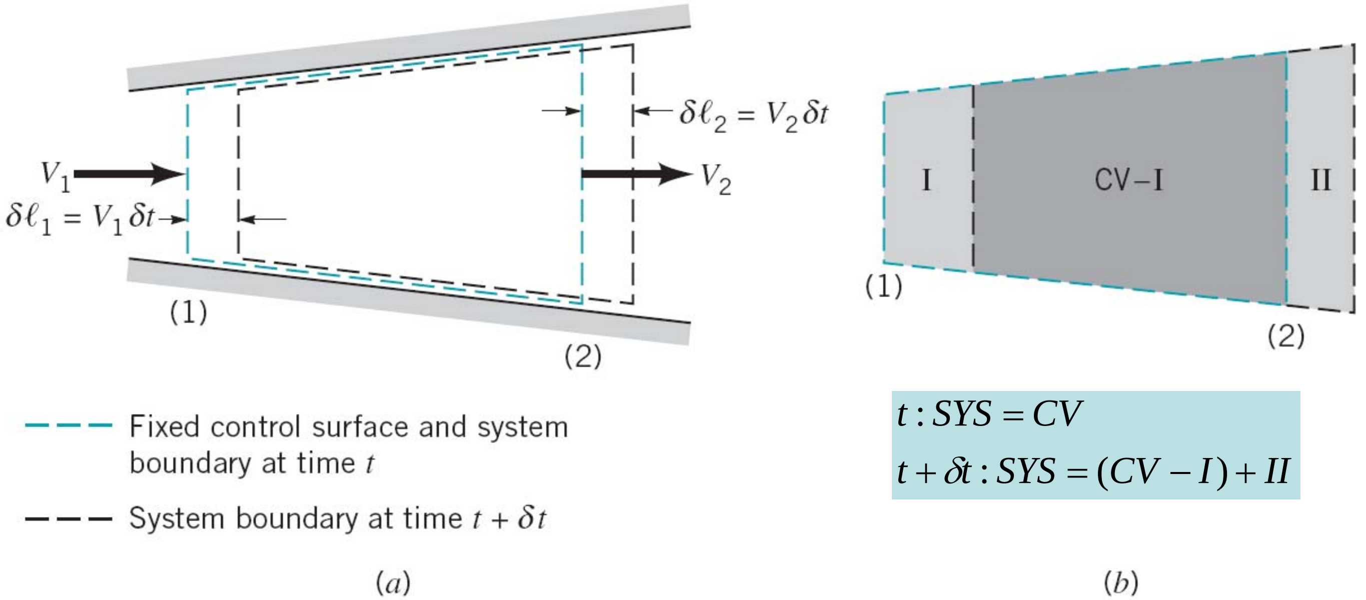
$$\frac{\partial B_{sys}}{\partial t} = \frac{\partial}{\partial t}\int_{sys} \rho b d\mathcal{V}$$

$$\frac{\partial B_{cv}}{\partial t} = \frac{\partial}{\partial t}\int_{cv} \rho b d\mathcal{V}$$



$$\frac{\partial B_{sys}}{\partial t} = \frac{\partial}{\partial t}\int_{sys} \rho d\mathcal{V} = 0; \frac{\partial B_{cv}}{\partial t} = \frac{\partial}{\partial t}\int_{cv} \rho d\mathcal{V} < 0$$

The Reynolds Transport theorem (simplified)



$t : SYS = CV$

$t + \delta t : SYS = (CV - I) + II$

Let's consider an extensive property B:

$$B_{sys}(t) = B_{cv}(t)$$

$$B_{sys}(t+\delta t) = B_{cv}(t+\delta t) - B_I(t+\delta t) + B_{II}(t+\delta t)$$

⇓

inflow outflow

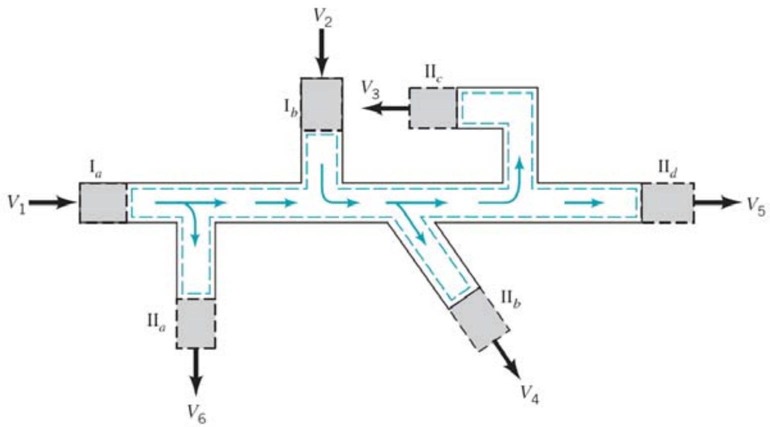
$$\frac{B_{sys}(t+\delta t) - B_{sys}(t)}{\delta t} = \frac{B_{cv}(t+\delta t) - B_{cv}(t)}{\delta t} - \frac{B_I(t+\delta t)}{\delta t} + \frac{B_{II}(t+\delta t)}{\delta t}$$

$$\frac{DB_{sys}}{Dt} = \frac{\partial B_{cv}}{\partial t} - \dot{B}_{in} + \dot{B}_{out}$$

- For fixed control volume with one inlet, one outlet, velocity normal to inlet/outlet

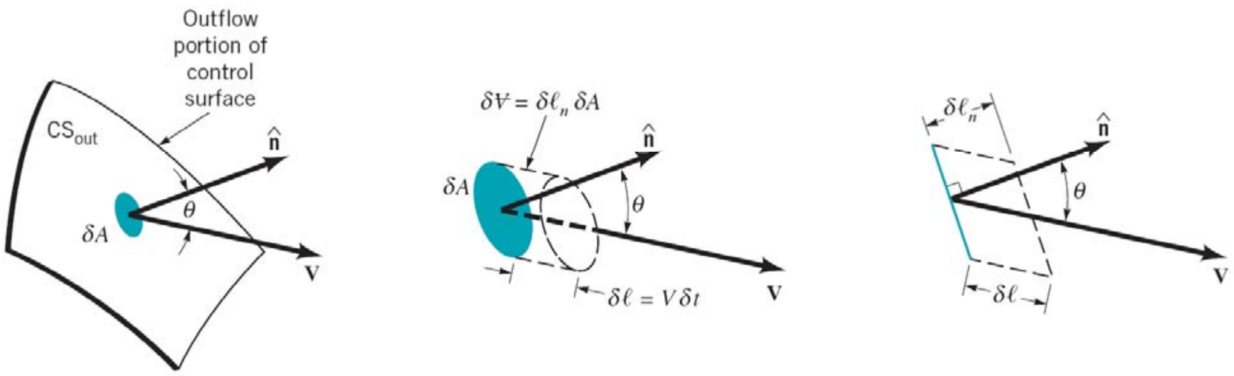
$$\frac{DB_{sys}}{Dt} = \frac{\partial B_{cv}}{\partial t} - \rho_1 A_1 V_1 b_1 + \rho_2 A_2 V_2 b_2$$

- Can be easily generalized:



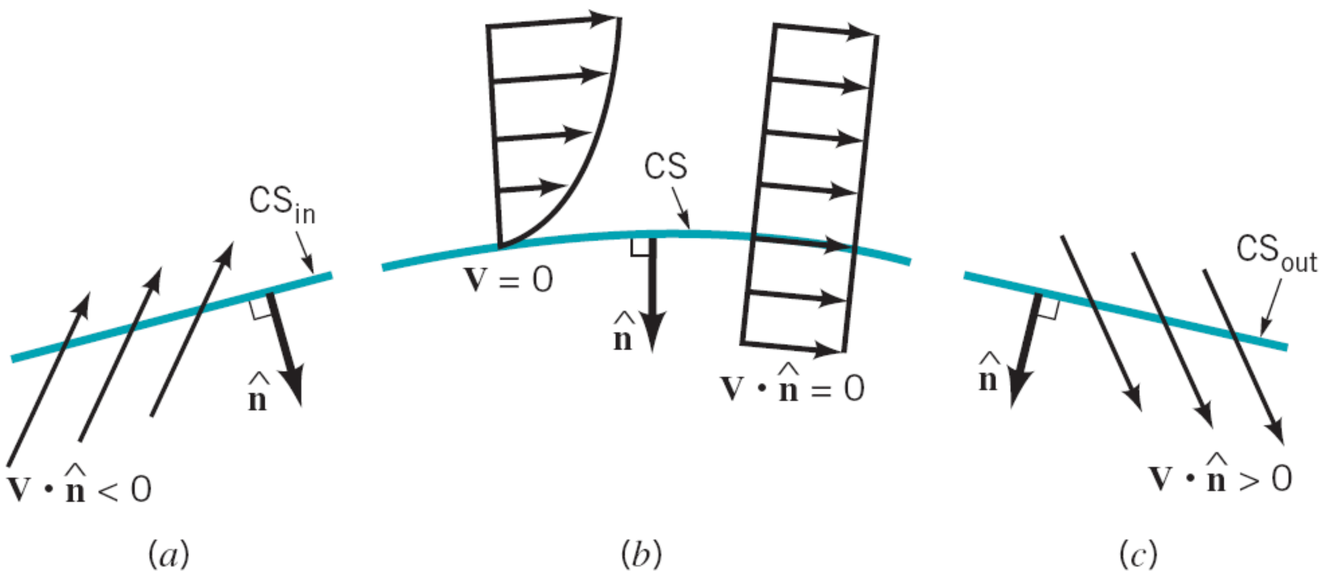
The Reynolds Transport theorem (for fixed nondeforming volume)

$$\delta B = b\rho\,\delta V\!\!\!\!-\; = b\rho(V\cos\theta\,\delta t)\,\delta A$$



(a) (b) (c)

$$\dot{B}_{out} = \int_{CV_{out}} \rho b \mathbf{V} \cdot \hat{\mathbf{n}} dA$$



$$\frac{DB_{Sys}}{Dt} = \frac{\partial B_{CV}}{\partial t} + \int_{CV} \rho b \mathbf{V} \cdot \hat{\mathbf{n}} dA$$

General Reynolds transport theorem for fixed control volume

Application of Reynolds Transport Theorem

$$\frac{DB_{Sys}}{Dt} = \frac{\partial B_{CV}}{\partial t} + \int_{CV} \rho b \mathbf{V} \cdot \widehat{\mathbf{n}} dA$$

- We will apply it now to various properties:
  - mass (continuity equation)
  - momentum (Newton 2nd law)
  - energy

Conservation of mass

- The amount of mass in the system should be conserved:

$$\frac{DM_{Sys}}{Dt} = \frac{D}{Dt}\int_{sys} \rho d\,\forall = \frac{\partial}{\partial t}\int_{CV} \rho d\,\forall + \int_{CV} \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA = 0$$

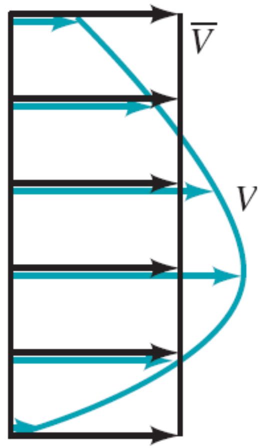
Continuity equation

Mass flow rate through a section of control surface having area A:

$$\dot{m} = \rho Q = \rho A \bar{V}$$

Volume flow rate

Average velocity: $\bar{V} = \dfrac{\int_A \rho \vec{V} \cdot \vec{n}\, dA}{\rho A}$



- For incompressible flow, the volume flowrate into a control volume equals the volume flowrate out of it.
- The overflow drain holes in a sink must be large enough to accommodate the flowrate from the faucet if the drain hole at the bottom of the sink is closed. Since the elevation head for the flow through the overflow drain is not large, the velocity there is relatively small. Thus, the area of the overflow drain holes must be larger than the faucet outlet area

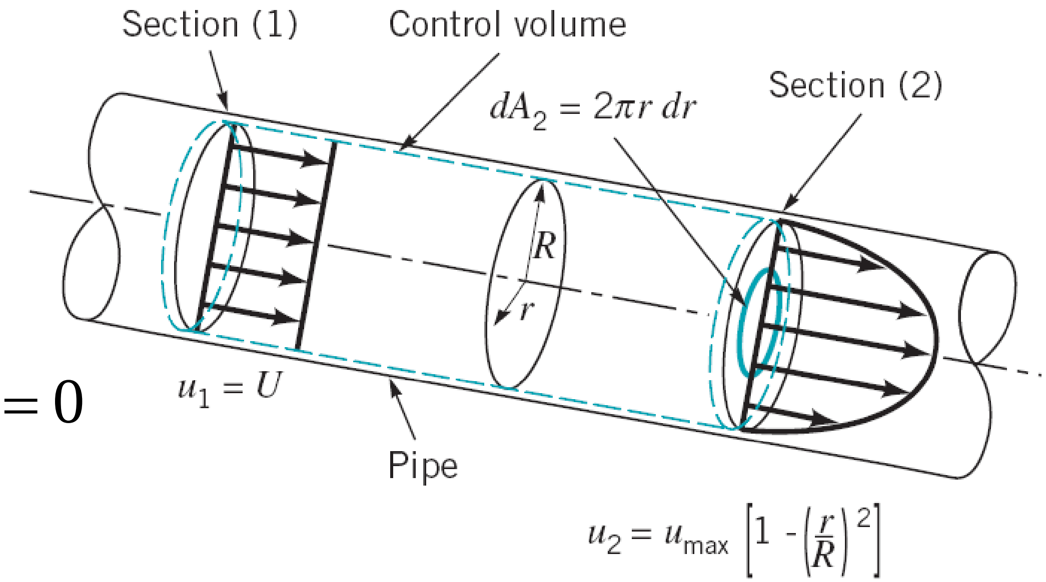
Example

Incompressible laminar flow develops in a straight pipe of radius R. At section 1 velocity profile is uniform, at section 2 profile is axisymmetric and parabolic with maximum value u_{max}. Find relation between U and u_{max}, what is average velocity at section (2)?

$$-\rho_1 A_1 U + \int_{A_2} \rho \vec{V} \cdot \vec{n} \, dA_2 = 0$$

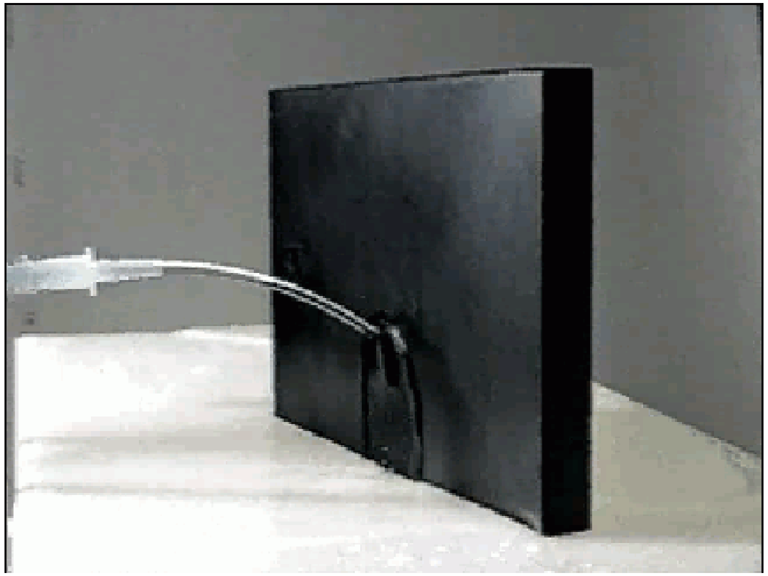
$$-A_1 U + u_{\max} \int_0^R \left[1 - \left(\frac{r}{R} \right)^2 \right] 2\pi r dr = 0$$

$$u_{\max} = 2U$$



Newton second law and conservation of momentum & momentum-of-momentum

A jet of fluid deflected by an object puts a force on the object. This force is the result of the change of momentum of the fluid and can happen even though the speed (magnitude of velocity) remains constant.



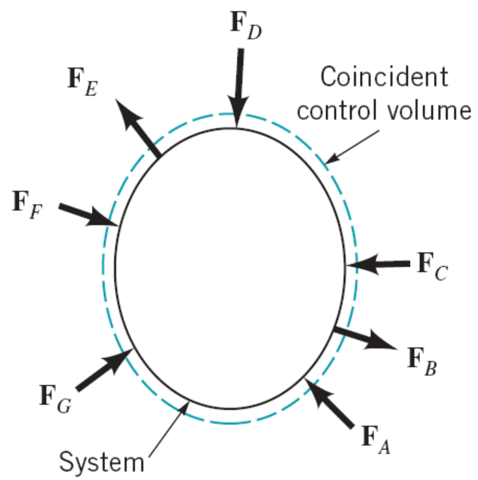
Newton second law and conservation of momentum & momentum-of-momentum

In an inertial coordinate system:

$$\frac{D}{Dt}\int_{sys} \mathbf{V}\rho d\forall = \sum F_{sys}$$

Rate of change of the momentum of the system

Sum of all external forces acting on the system



At a moment when system coincide with control volume:

$$\sum F_{sys} = \sum F_{\substack{\text{contents of} \\ \text{the control volume}}}$$

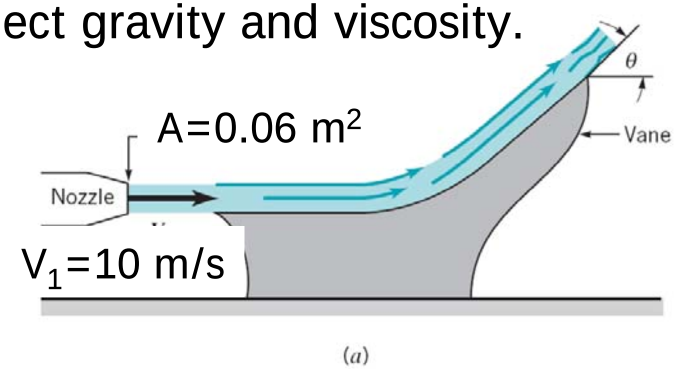
On the other hand: $$\frac{D}{Dt}\int_{sys} \mathbf{V}\rho d\forall = \frac{\partial}{\partial t}\int_{CV} \mathbf{V}\rho d\forall + \int_{CS} \mathbf{V}\rho \mathbf{V}\cdot\hat{\mathbf{n}} dA$$

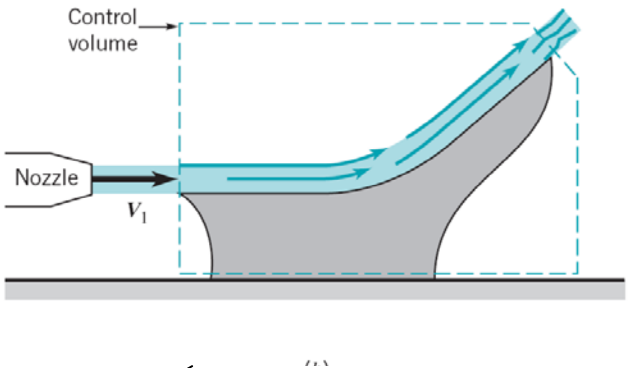
$$\frac{\partial}{\partial t}\int_{CV} \mathbf{V}\rho d\forall + \int_{CS} \mathbf{V}\rho \mathbf{V}\cdot\hat{\mathbf{n}} dA = \sum F_{\substack{\text{contents of} \\ \text{the control volume}}}$$

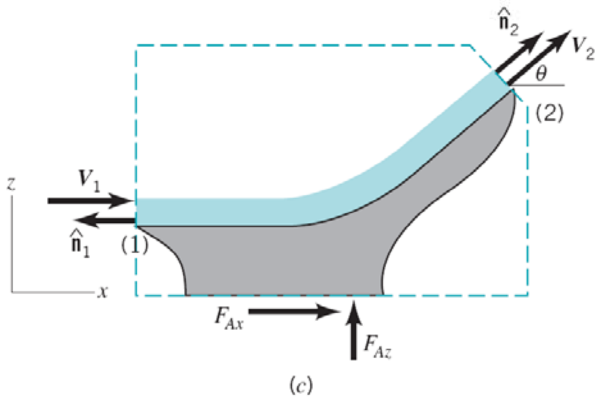
Example: Linear momentum

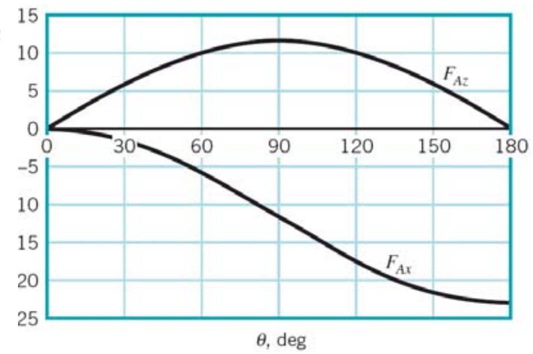
Determine anchoring forces required to keep the vane stationary vs angle Θ.
Neglect gravity and viscosity.

■ FIGURE E5.10a-c









$$\frac{\partial}{\partial t}\int_{CV} u\rho d\forall + \int_{CS} \rho u \mathbf{V}\cdot\hat{\mathbf{n}} dA = \sum F_x$$

$$\frac{\partial}{\partial t}\int_{CV} w\rho d\forall + \int_{CS} \rho w \mathbf{V}\cdot\hat{\mathbf{n}} dA = \sum F_y$$

$$\Rightarrow$$

$$V_1\rho(-V_1)A_1 + V_1\cos\theta\rho(V_1)A_2 = F_{Ax}$$

$$0\rho(-V_1)A_1 + V_1\sin\theta\rho(V_1)A_2 = F_{Az}$$

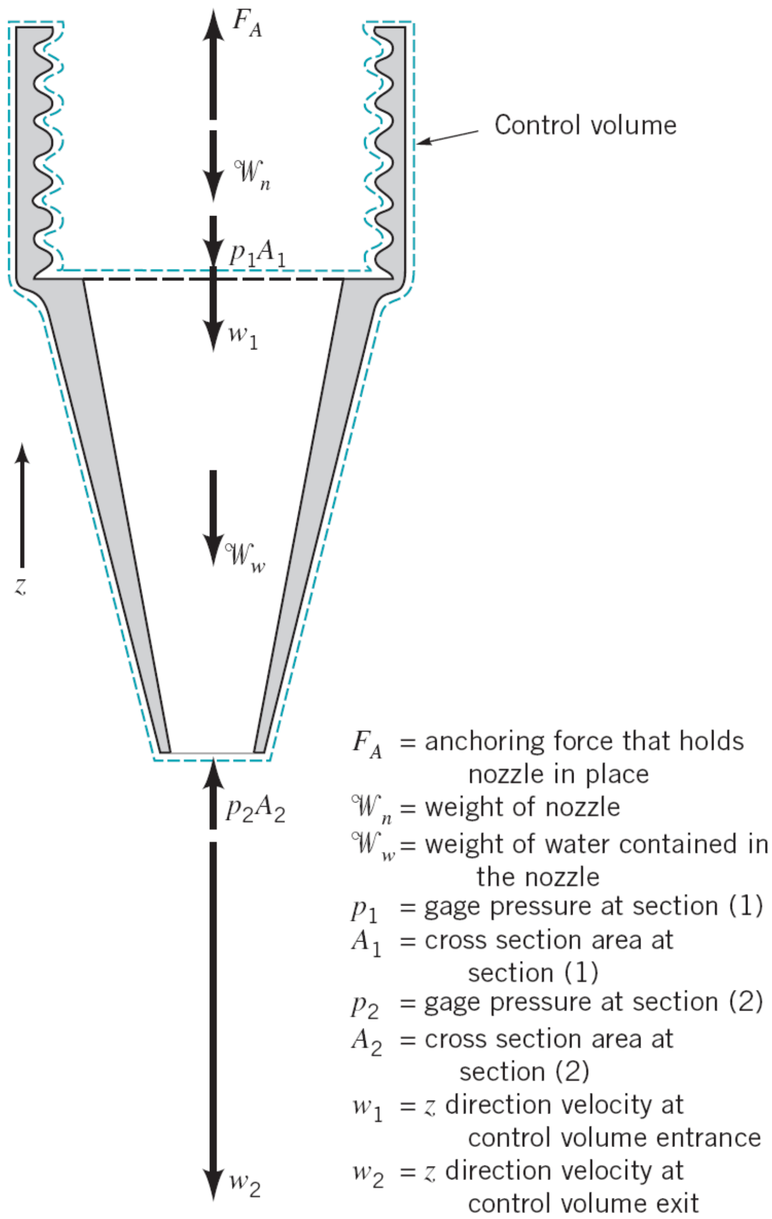
$$\Downarrow$$

$$F_{Ax} = -V_1^2\rho A_1(1-\cos\theta)$$

$$F_{Az} = V_1^2\rho A_1\sin\theta$$

Linear momentum: comments

- Linear momentum is a vector
- As normal vector points outwards, momentum flow inside a CV involves negative **Vn** product and moment flow outside of a CV involves a positive **Vn** product.
- The time rate of change of the linear momentum of the contents of a nondeforming CV is zero for steady flow
- Forces due to atmospheric pressure on the CV may need to be considered



F_A
$\mathcal{W}_n$
p_1A_1
w_1
Control volume
z
$\mathcal{W}_w$
p_2A_2
w_2
F_A = anchoring force that holds nozzle in place
$\mathcal{W}_n$ = weight of nozzle
$\mathcal{W}_w$ = weight of water contained in the nozzle
p_1 = gage pressure at section (1)
A_1 = cross section area at section (1)
p_2 = gage pressure at section (2)
A_2 = cross section area at section (2)
w_1 = z direction velocity at control volume entrance
w_2 = z direction velocity at control volume exit

Moment-of-Momentum Equation

The net rate of flow of moment-of-momentum through a control surface equals the net torque acting on the contents of the control volume.

Water enters the rotating arm of a lawn sprinkler along the axis of rotation with no angular momentum about the axis. Thus, with negligible frictional torque on the rotating arm, the absolute velocity of the water exiting at the end of the arm must be in the radial direction (i.e., with zero angular momentum also). Since the sprinkler arms are angled "backwards", the arms must therefore rotate so that the circumferential velocity of the exit nozzle (radius times angular velocity) equals the oppositely directed circumferential water velocity.

The Energy Equation

$$\frac{D}{Dt}\int_{sys} e\rho d\forall = (\dot{Q} + \dot{W})_{sys}$$

Rate of increase of the total stored energy of the system

Net rate of energy addition by heat transfer into the system

Net rate of energy addition by work transfer into the system

Total stored energy per unit mass:

$$e = u + \frac{V^2}{2} + gz$$

$$\frac{D}{Dt}\int_{sys} e\rho d\forall = \frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS} e\rho \mathbf{V}\cdot\hat{\mathbf{n}} dA$$

Rate of increase of the total stored energy of the system

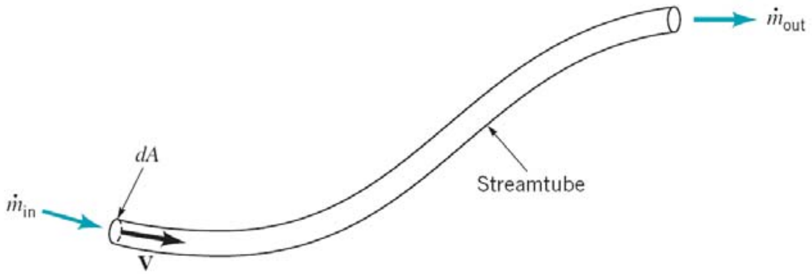
rate of increase of the total stored energy of the control volume

Net rate of energy flow out of the control volume

$$\frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS} e\rho \mathbf{V}\cdot\hat{\mathbf{n}} dA = (\dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{net\\in}})_{cv}$$

Application of energy equation

- Product **V·n** is non-zero only where liquid crosses the CS; if we have only one stream entering and leaving control volume:



$$\int_{CS}\left(\breve{u}+\frac{p}{\rho}+\frac{V^2}{2}+gz\right)\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA=\left(\breve{u}+\frac{p}{\rho}+\frac{V^2}{2}+gz\right)_{out}\dot{m}_{out}-\left(\breve{u}+\frac{p}{\rho}+\frac{V^2}{2}+gz\right)_{in}\dot{m}_{in}$$

- If no shaft power is applied and we assume flow steady

$$\underbrace{\frac{p_{out}}{\rho}+\frac{V_{out}^{\ 2}}{2}+gz_{out}}_{\textbf{\textit{available energy}}}=\frac{p_{in}}{\rho}+\frac{V_{in}^{\ 2}}{2}+gz_{in}-\underbrace{(\breve{u}_{out}-\breve{u}_{in}-q_{\substack{net\\in}})}_{\textbf{loss}},\qquad q_{\substack{net\\in}}=\frac{\dot{Q}_{\substack{net\\in}}}{\dot{m}}$$

available energy **loss**

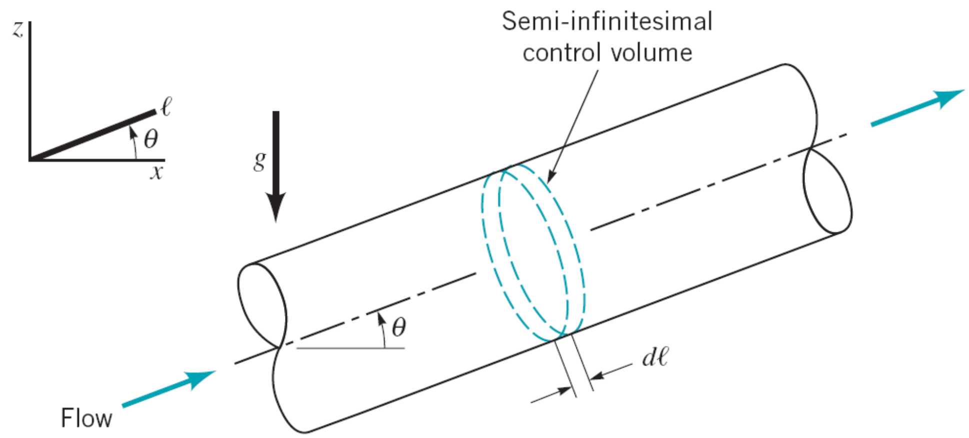
Energy transfer

Work must be done on the device shown to turn it over because the system gains potential energy as the heavy (dark) liquid is raised above the light (clear) liquid. This potential energy is converted into kinetic energy which is either dissipated due to friction as the fluid flows down the ramp or is converted into power by the turbine and then dissipated by friction. The fluid finally becomes stationary again. The initial work done in turning it over eventually results in a very slight increase in the system temperature

Second law of thermodynamics

- Let's apply "stream line energy equation" to an infinitesimally thin volume

$$\dot{m}\left[d\breve{u} + d\left(\frac{p}{\rho}\right) + d\left(\frac{V^2}{2}\right) + g\,dz\right] = \delta\dot{Q}_{\substack{net\\in}}$$



For closed system in the absence of additional work:

$$dU = TdS - pdV \quad \Rightarrow \quad d\breve{u} = Tds - pd\left(\frac{1}{\rho}\right)$$

$$\left[\frac{dp}{\rho} + d\left(\frac{V^2}{2}\right) + g\,dz\right] = -\left(Tds - \delta\dot{q}_{\substack{net\\in}}\right)$$

If we take into account Clausius inequality: $dS - \frac{dq}{T} \geq 0$

$$-\left[\frac{dp}{\rho} + d\left(\frac{V^2}{2}\right) + g\,dz\right] \geq 0$$

Application of energy equation

- We can make the energy equation more concrete by noting:
  - Work is usually transferred into liquid by rotating shaft:

$$\dot{\boldsymbol{W}}_{shaft} = T_{shaft}\omega$$

  - Or by normal stress acting on a free surface

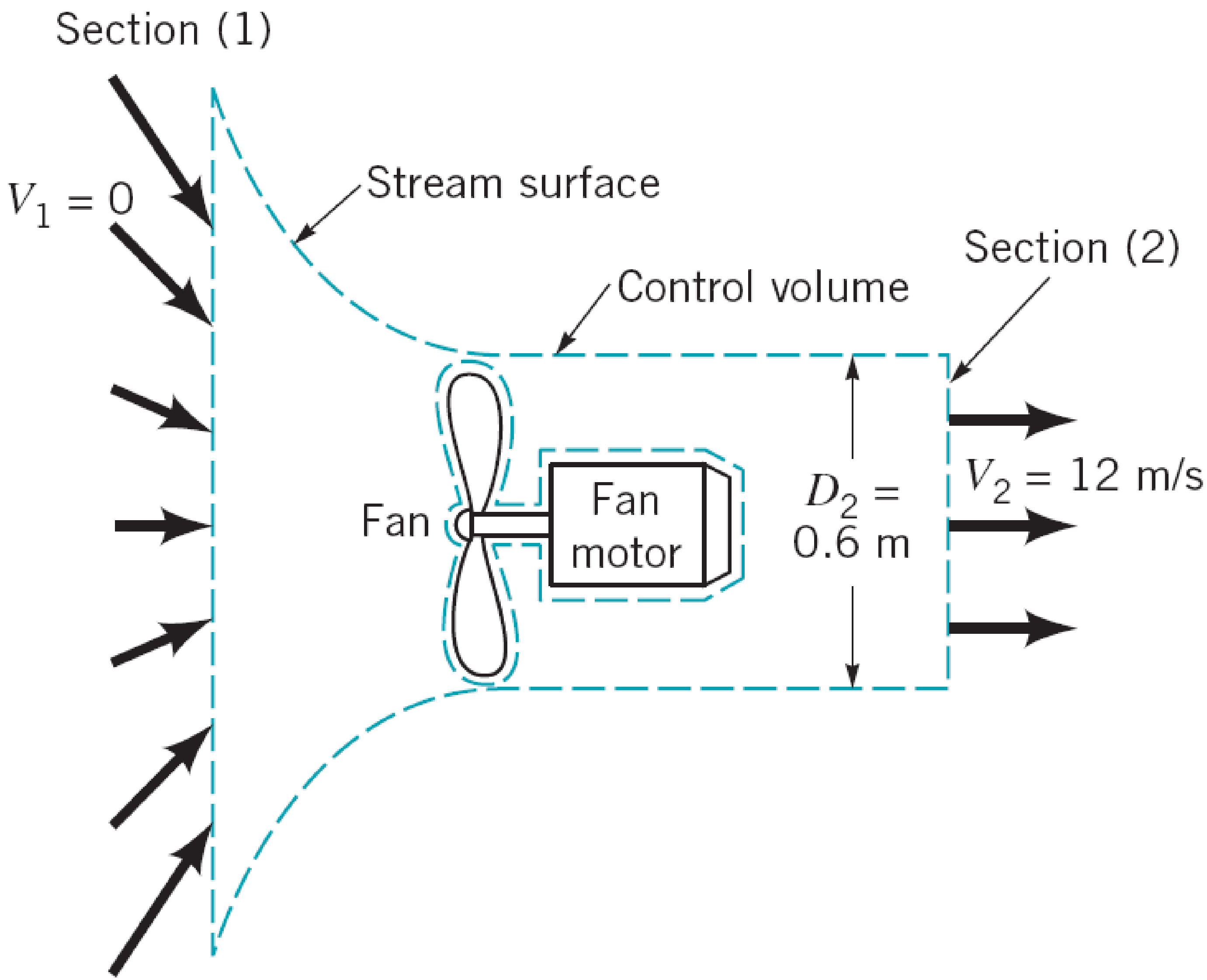
$$\delta\dot{\boldsymbol{W}}_{normal\,stress} = \sigma\hat{\boldsymbol{n}}\delta A\cdot\boldsymbol{V} = -p\boldsymbol{V}\cdot\hat{\boldsymbol{n}}\delta A$$

$$\dot{\boldsymbol{W}}_{normal\,stress} = \int_{\mathrm{CS}} -p\boldsymbol{V}\cdot\hat{\boldsymbol{n}}\,dA$$

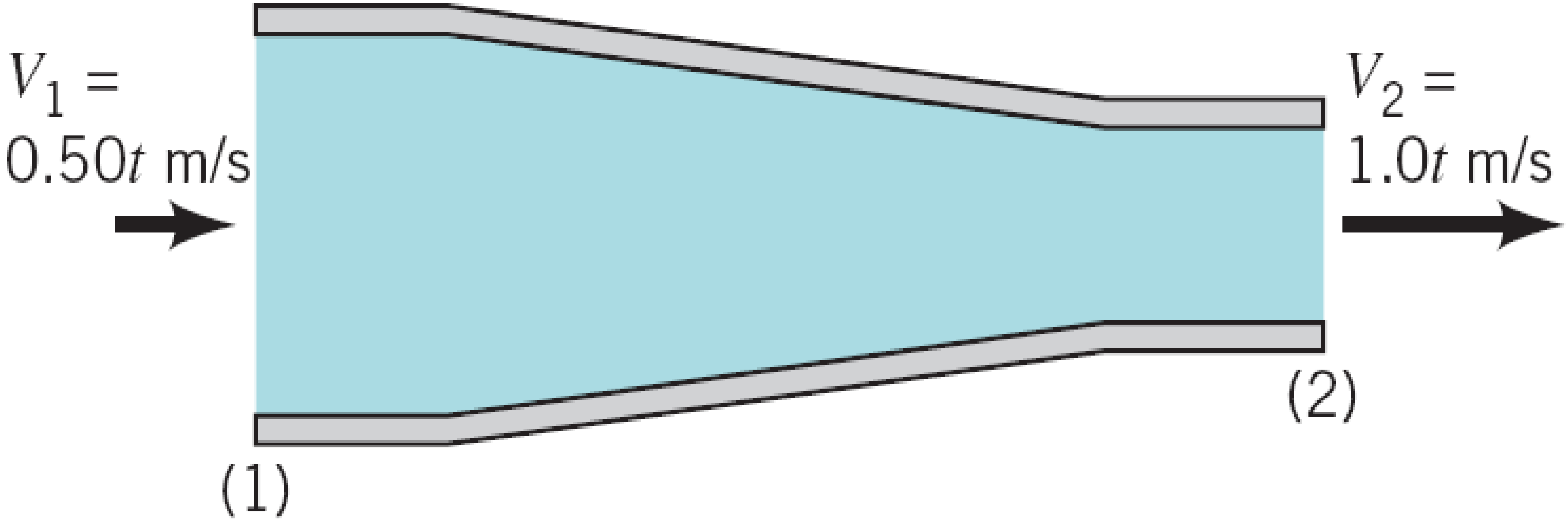
- The energy equation is now:

$$\frac{\partial}{\partial t}\int_{CV} e\rho d\forall + \int_{CS}\left(\breve{u} + \frac{p}{\rho} + \frac{V^2}{2} + gz\right)\rho\mathbf{V}\cdot\hat{\mathbf{n}}dA = (\dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{shaft\\in}})_{cv}$$

Example: Efficiency of a fan



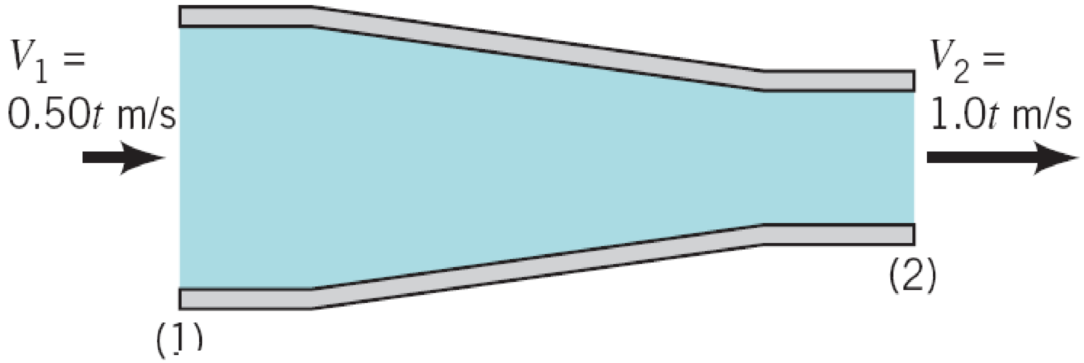
Problem 4.20



- Determine local acceleration at points 1 and 2. Is the average convective acceleration between these points negative, zero or positive?

Problems

- **4.20.** Determine local acceleration at points 1 and 2. Is the average convective acceleration between these points negative, zero or positive?



- **5.102** Water flows steadily down the inclined pump. Determine:
  - The pressure difference, p_1-p_2;
  - The loss between sections 1 and 2
  - The axial force exerted on the pipe by water