Lecture 5

Differential Analysis of Fluid Flow

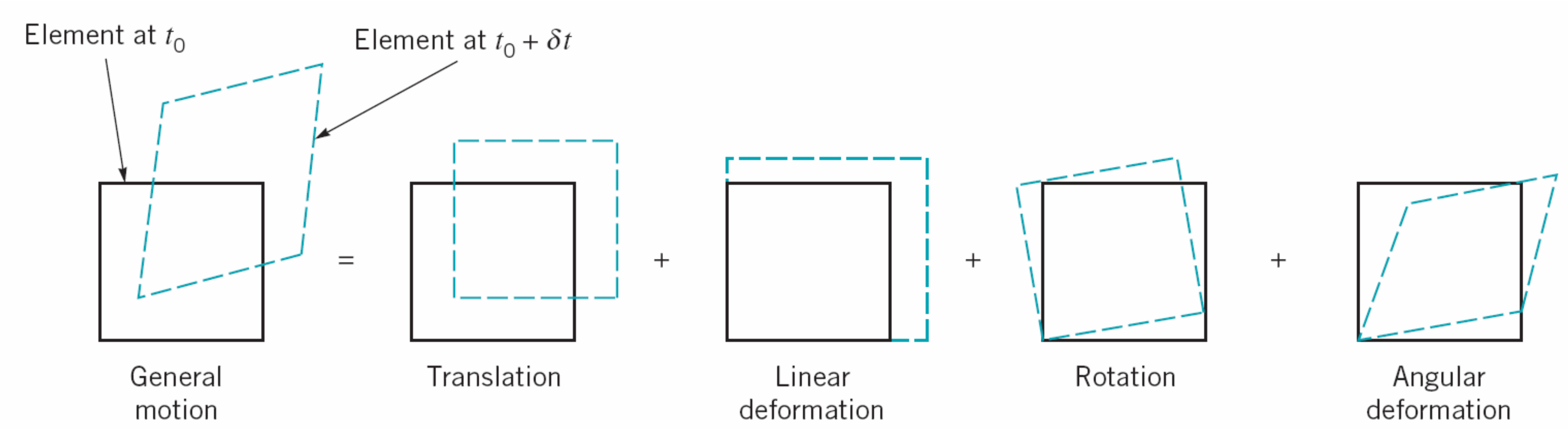
Navier-Stockes equation

Differential analysis of Fluid Flow

- The aim: to produce differential equation describing the motion of fluid in detail

Fluid Element Kinematics

- Any fluid element motion can be represented as consisting of translation, linear deformation, rotation and angular deformation



Velocity and acceleration field

- Velocity field

$$\boldsymbol{V} = u\hat{\boldsymbol{i}} + v\hat{\boldsymbol{j}} + w\hat{\boldsymbol{k}}$$

- Acceleration

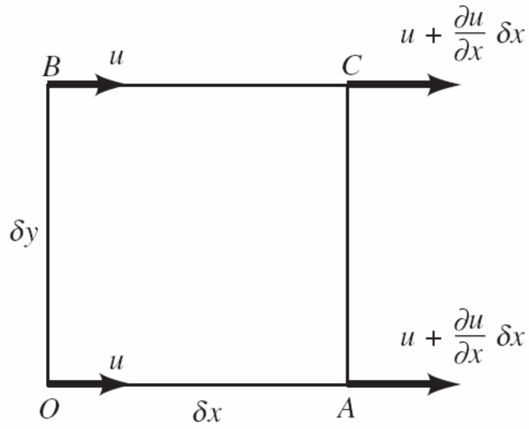
$$\mathbf{a}(r,t) = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = \frac{D\mathbf{V}}{Dt}$$

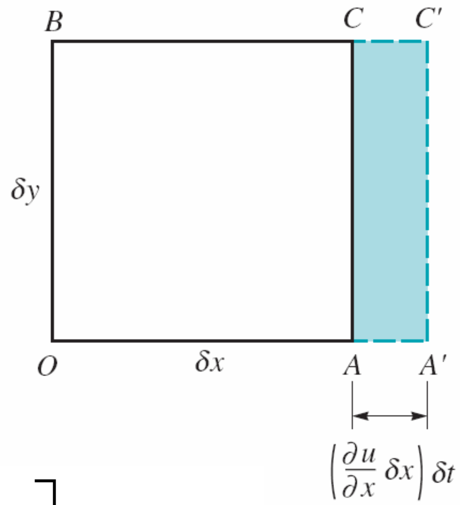
- Material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + (\mathbf{V} \bullet \nabla)$$

Linear motion and deformation

- Let's consider stretching of a fluid element under velocity gradient in one direction



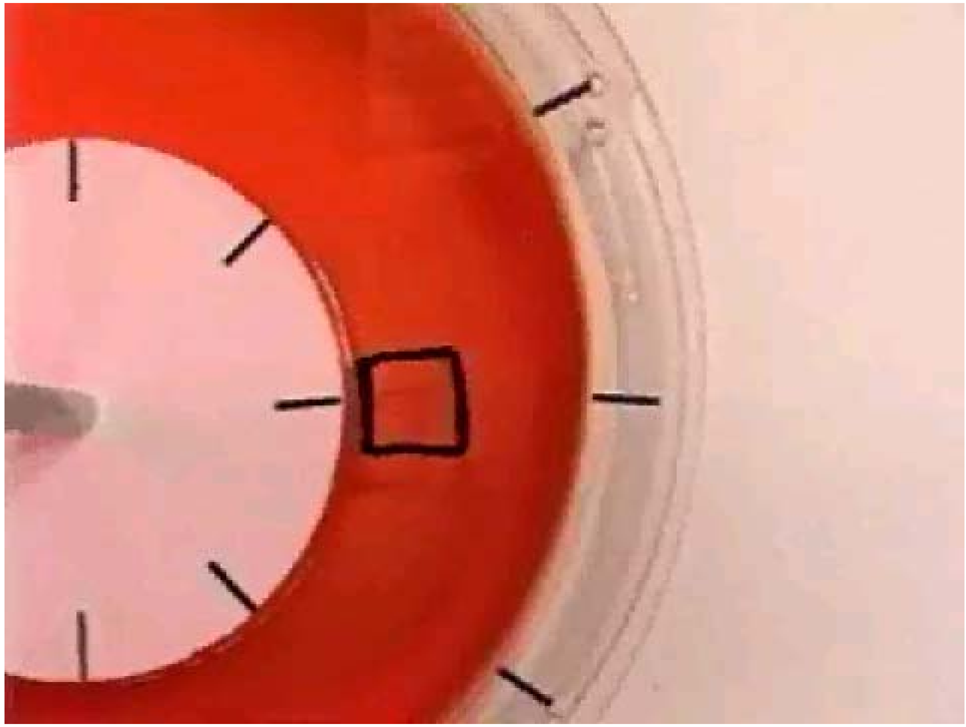


<u>Volumetric dilatation rate:</u>

$$\frac{1}{\delta \mathcal{V}}\frac{d(\delta \mathcal{V})}{dt} = \lim_{\delta t \to 0}\left[\frac{\left(\frac{\partial u}{\partial x}\right)\delta x \delta t\, \delta y \delta z}{\delta x \delta y \delta z\, \delta t}\right] = \left(\frac{\partial u}{\partial x}\right)$$

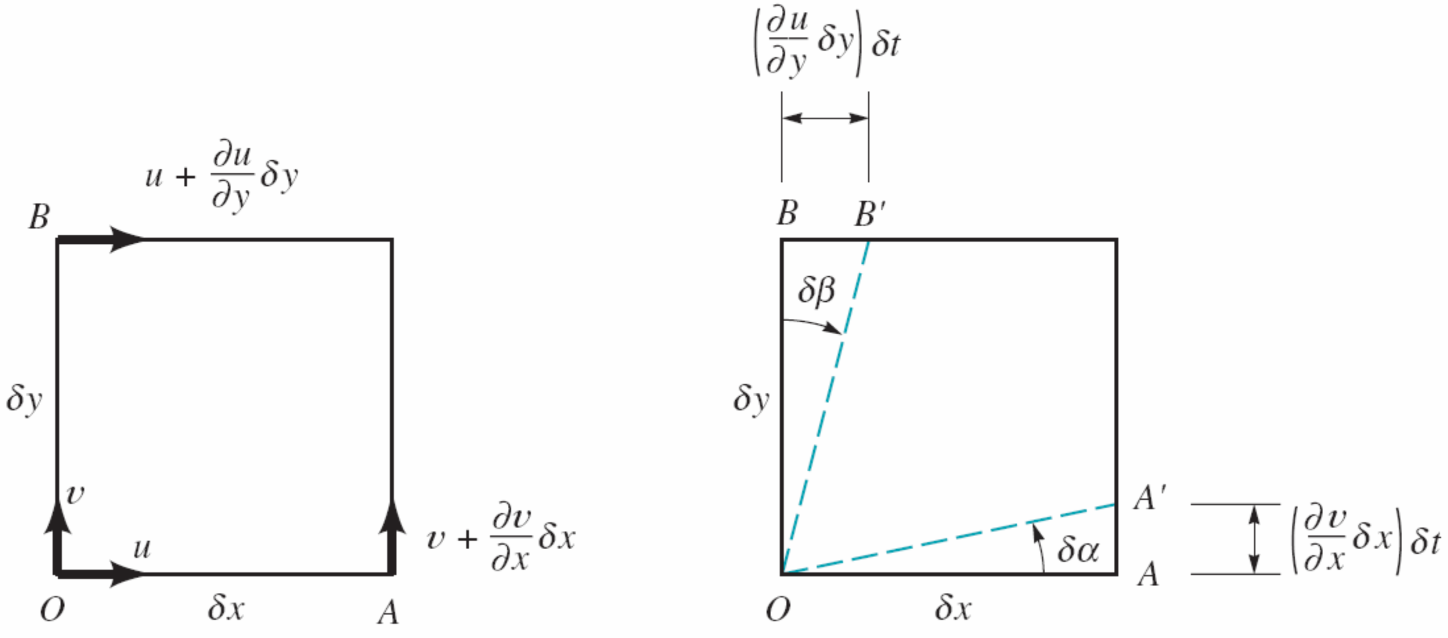
$$\frac{1}{\delta \mathcal{V}}\frac{d(\delta \mathcal{V})}{dt} = \left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial z}\right) = \nabla \cdot \boldsymbol{V}$$

Angular motion and deformation



Fluid elements located in a moving fluid move with the fluid and generally undergo a change in shape (angular deformation).
A small rectangular fluid element is located in the space between concentric cylinders. The inner wall is fixed. As the outer wall moves, the fluid element undergoes an angular deformation. The rate at which the corner angles change (rate of angular deformation) is related to the shear stress causing the deformation

Angular motion and deformation



(b)

$$\omega_{OA} = \lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t} \approx \frac{(\partial v / \partial x)\delta x \delta t}{\delta x}\frac{1}{\delta t} = \frac{\partial v}{\partial x}$$

$$\omega_{OB} = \frac{\partial u}{\partial y}$$

- Rotation is defined as the average of those velocities:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Angular motion and deformation

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\boldsymbol{\omega} = \frac{1}{2}\nabla \times \boldsymbol{V} = \frac{1}{2}\begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix} =$$

$$= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\boldsymbol{i}} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\boldsymbol{j}} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\boldsymbol{k}}$$

- Vorticity is defined as twice the rotation vector

$$\zeta = 2\omega = \nabla \times \boldsymbol{V}$$

- If rotation (and vorticity) is zero flow is called **irrotational**

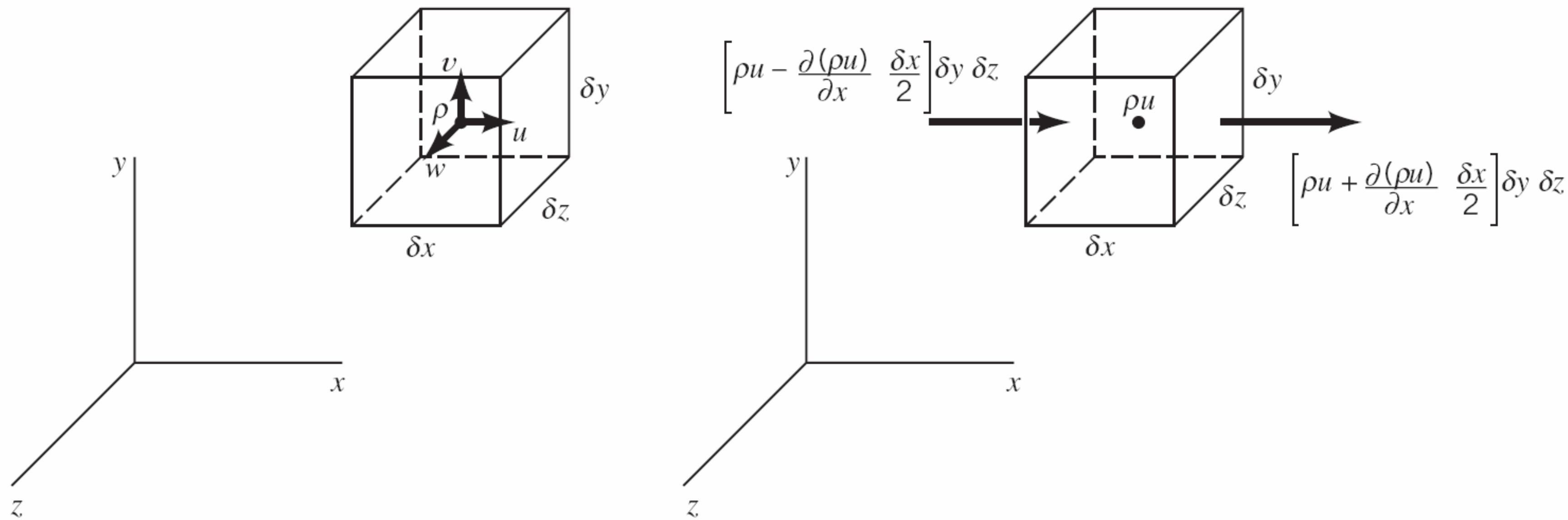
Angular motion and deformation

- Rate of shearing strain (or rate of angular deformation) can be defined as sum of fluid element rotations:

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Conservation of mass

- As we found before: $$\frac{DM_{Sys}}{Dt} = \frac{\partial}{\partial t}\int_{CV} \rho d\forall + \int_{CV} \rho \mathbf{V}\cdot\hat{\mathbf{n}}dA = 0$$



$$\frac{\partial}{\partial t}\int_{CV} \rho d\forall = \frac{\partial \rho}{\partial t}\delta x \delta y \delta z$$

Flow of mass in x-direction: $$\frac{\partial(\rho u)}{\partial x}\delta x \delta y \delta z$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad \frac{\partial \rho}{\partial t} + \nabla\cdot\rho\mathbf{V} = 0$$

Conservation of mass

- Incompressible flow

$$\nabla \cdot \boldsymbol{V} = 0 \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

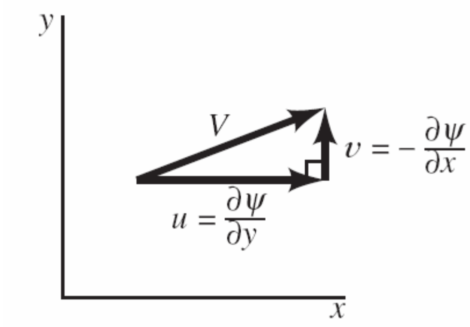
- Flow in cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r\rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

- Incompressible flow in cylindrical coordinates

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (v_\theta)}{\partial \theta} + \frac{\partial (v_z)}{\partial z} = 0$$

Stream function



- 2D incompressible flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

- We can define a scalar function such that

$$u = \frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \psi}{\partial x}$$

Stream function

- Lines along which stream is const are stream lines:

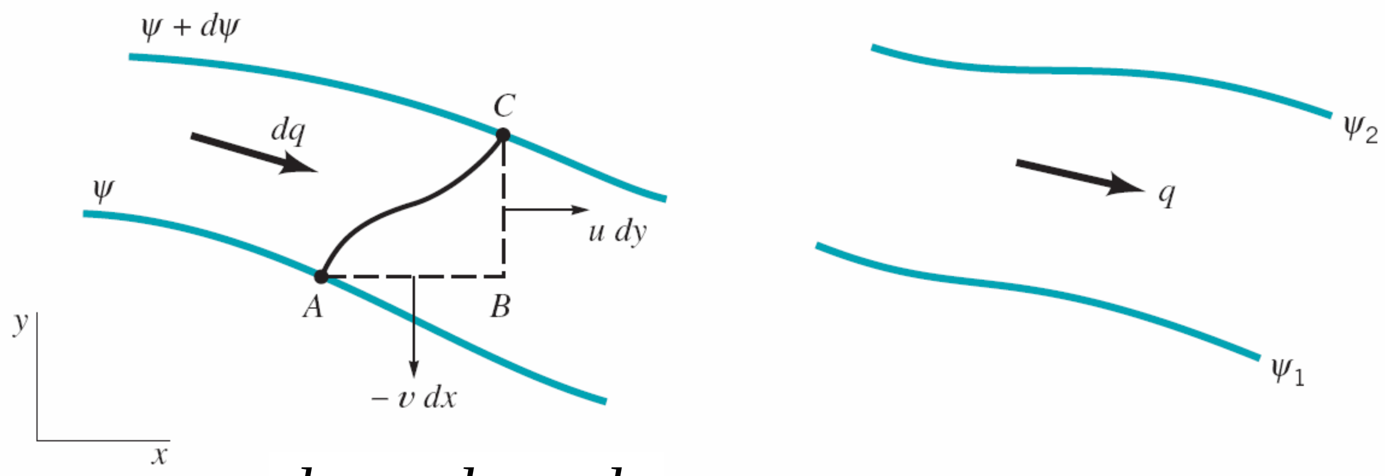
$$\frac{dy}{dx} = \frac{v}{u}$$

Indeed:

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -vdx + udy = 0$$

Stream function

- Flow between streamlines



$$dq = udy - vdx$$

(b)

$$\boxed{dq = \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = d\psi}$$

$$\boxed{q = \psi_2 - \psi_1}$$

Description of forces

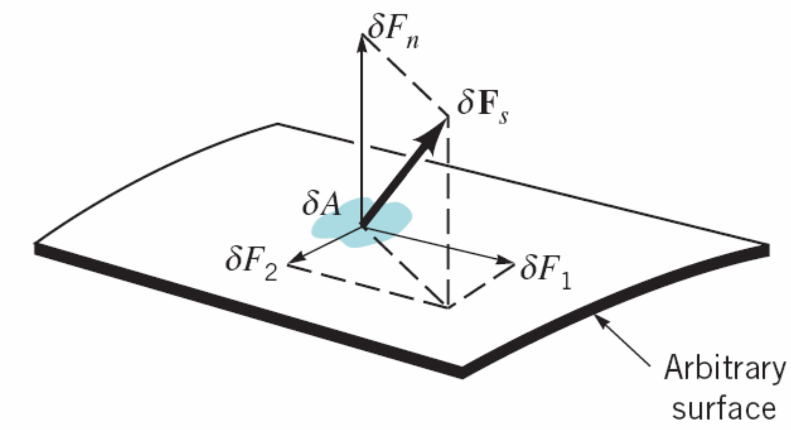
Forces

Body forces – distributed through the element, e.g. Gravity

$$\delta \boldsymbol{F} = \delta m \boldsymbol{g}$$

Surface forces – result of interaction with the surrounding elements: e.g. Stress

Normal stress



Shearing stresses

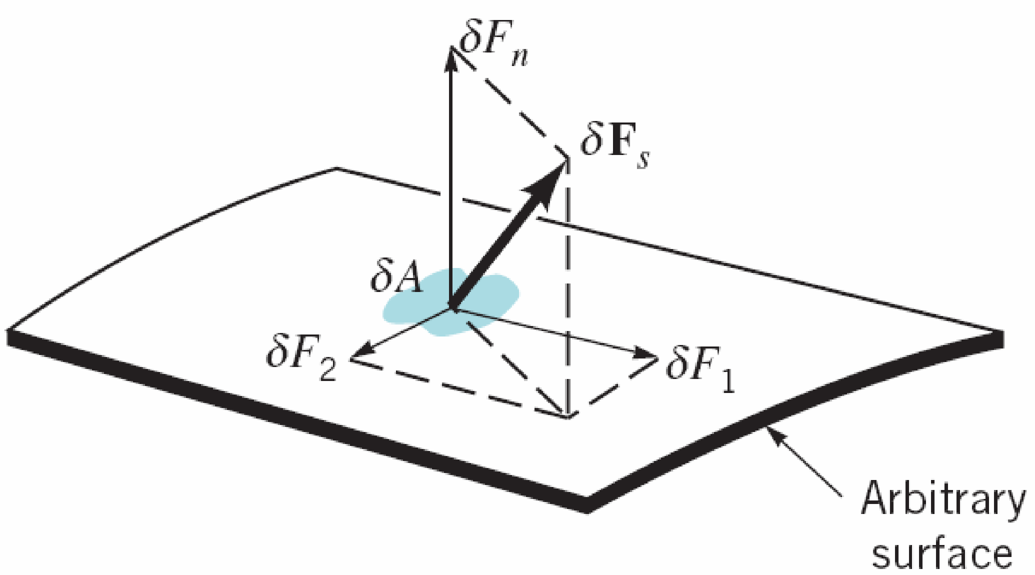
Linear forces: Surface tension

Stress acting on a fluidic element

- normal stress $$\sigma_n = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A}$$

- shearing stresses

$$\tau_1 = \lim_{\delta A \to 0} \frac{\delta F_1}{\delta A} \qquad \tau_2 = \lim_{\delta A \to 0} \frac{\delta F_2}{\delta A}$$



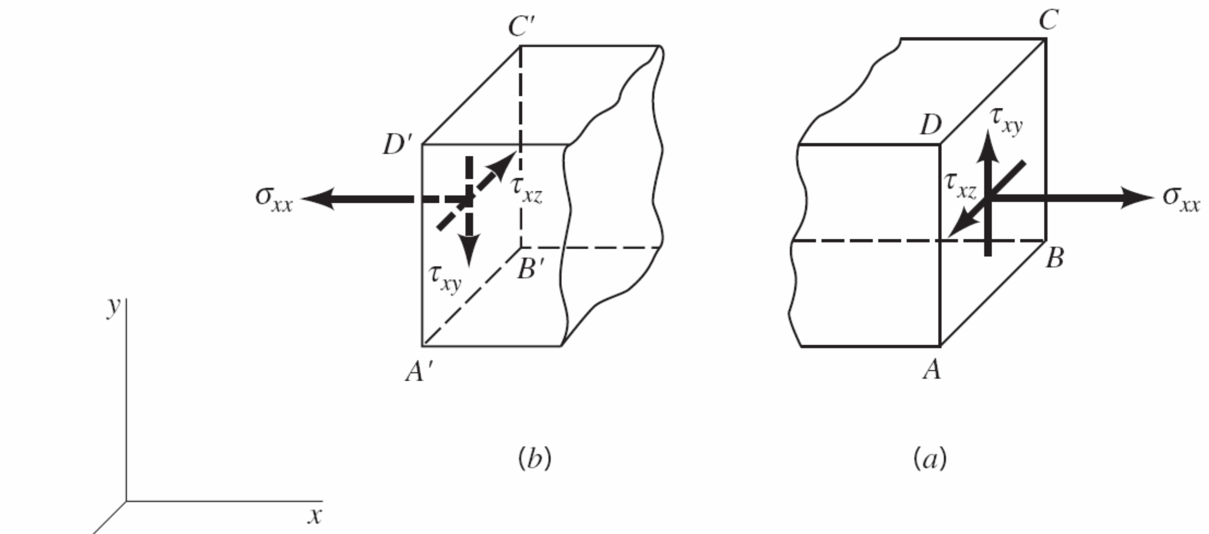
Stresses: double subscript notation

- normal stress: σ_{xx}

- shearing stress: τ_{xy} τ_{xz}

normal to the plane

direction of stress



sign convention: positive stress is directed in positive axis directions if surface normal is pointing in the positive direction

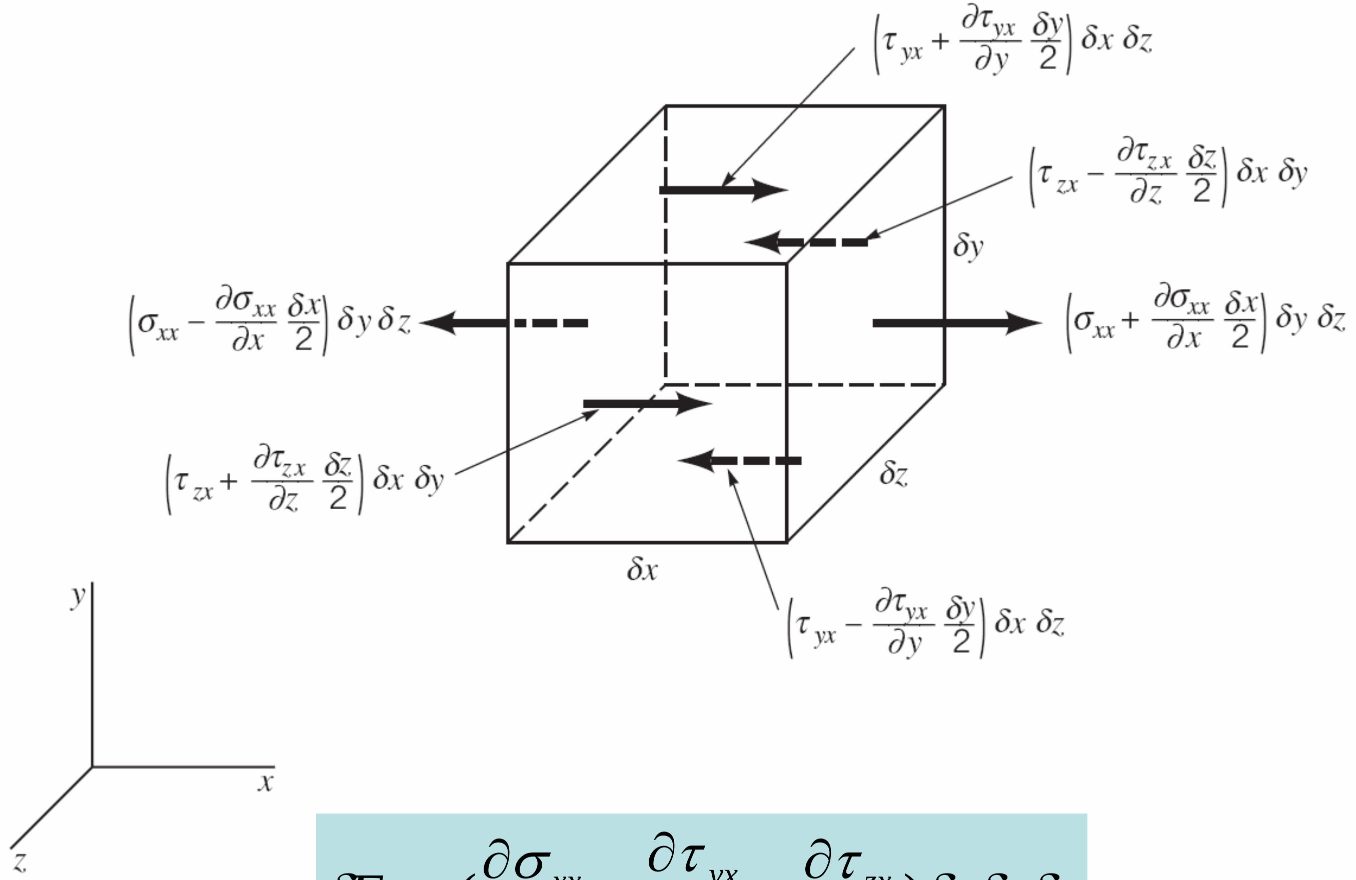
Vectors and Tensors

- To define stress at a point we need to define “stress vector” for all 3 perpendicular planes passing through the point

$$\tau = \begin{pmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{pmatrix}$$

Force on a fluid element

- To find force in each direction we need to sum all forces (normal and shearing) acting in the same direction



$$\delta F_x = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x \delta y \delta z$$

Differential equation of motion

$$\boxed{\delta\mathbf{F} = \delta m\mathbf{a}} \qquad \begin{aligned} \delta F_x &= \delta m a_x \\ \delta F_y &= \delta m a_y, \qquad \delta m = \delta x\,\delta y\,\delta z \\ \delta F_z &= \delta m a_z \end{aligned}$$

$$\rho g_x + \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) = \underbrace{\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)}_{\text{Acceleration (“material derivative”)}}$$

$$\rho g_y + \left(\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z}\right) = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z + \left(\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{xz}}{\partial x}\right) = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

Viscous Flow

$$\sigma_{xx} = -p + 2\mu \frac{du}{dx}$$

$$\sigma_{yy} = -p + 2\mu \frac{dv}{dy}$$

$$\sigma_{zz} = -p + 2\mu \frac{dw}{dz}$$

$$\tau_{xy} = \tau_{yx} = \mu(\frac{du}{dy} + \frac{dv}{dx})$$

$$\tau_{yz} = \tau_{zy} = \mu(\frac{dv}{dz} + \frac{dw}{dy})$$

$$\tau_{zx} = \tau_{xz} = \mu(\frac{du}{dz} + \frac{dw}{dx})$$

for viscous flow normal
stresses are not necessary the
same in all directions

Navier-Stokes Equations

$$\rho(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z})=-\frac{\partial p}{\partial x}+\rho g_x+\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})$$

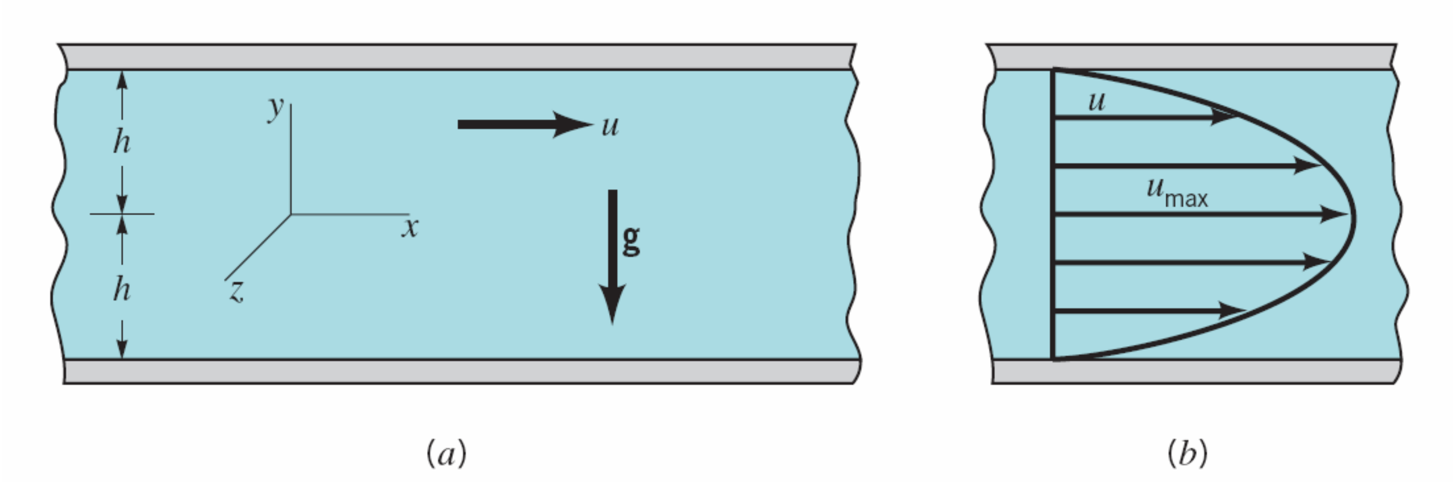
$$\rho(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z})=-\frac{\partial p}{\partial y}+\rho g_y+\mu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2})$$

$$\rho(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z})=-\frac{\partial p}{\partial z}+\rho g_z+\mu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

- 4 equations for 4 unknowns (*u*,*v*,*w*,*p*)
- Analytical solution are known for only few cases

Steady Laminar Flow between parallel plates



$$\rho(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z})=-\frac{\partial p}{\partial x}+\rho g_x+\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})$$

$$\rho(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z})=-\frac{\partial p}{\partial y}+\rho g_y+\mu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2})$$

$$\rho(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z})=-\frac{\partial p}{\partial z}+\rho g_z+\mu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$v=0; w=0 \Rightarrow \frac{\partial u}{\partial x}=0$$

$$0=-\frac{\partial p}{\partial x}+\mu(\frac{\partial^2 u}{\partial y^2})$$

$$0=-\frac{\partial p}{\partial y}-\rho g$$

$$0=-\frac{\partial p}{\partial z}$$

$$v = 0; w = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$$

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$$0 = -\frac{\partial p}{\partial y} - \rho g$$

$$0 = -\frac{\partial p}{\partial z}$$

$$v = 0; w = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$$

$$\frac{du}{dy} = \frac{1}{\mu}\frac{\partial p}{\partial x} y + C_1 \Rightarrow u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + C_1 y + C_2$$

$$p = -\rho g y + f(x)$$

Boundary condition (no slip) $u(h) = u(-h) = 0$

Velocity profile $$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - h^2)$$

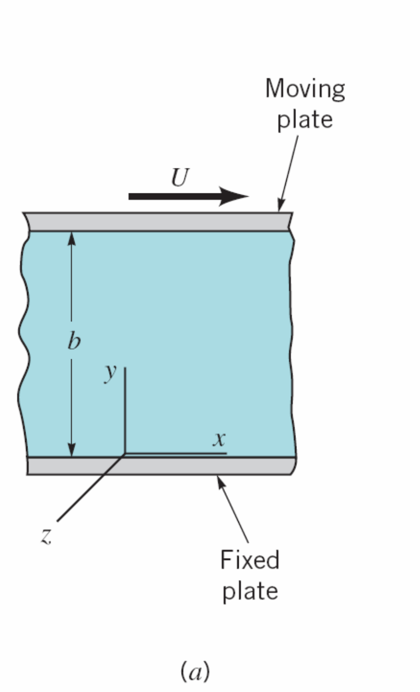
Flow rate $$q = \int_{-h}^{h} u\,dy = \int_{-h}^{-h} \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - h^2)\,dy = -\frac{2h^3}{3\mu}\left(\frac{\partial p}{\partial x}\right)$$

What is maximum velocity (u_{max}) and average velocity?

No-slip boundary condition

- Boundary conditions are needed to solve the differential equations governing fluid motion. One condition is that any viscous fluid sticks to any solid surface that it touches.
- Clearly a very viscous fluid sticks to a solid surface as illustrated by pulling a knife out of a jar of honey. The honey can be removed from the jar because it sticks to the knife. This no-slip boundary condition is equally valid for small viscosity fluids. Water flowing past the same knife also sticks to it. This is shown by the fact that the dye on the knife surface remains there as the water flows past the knife.

Couette flow



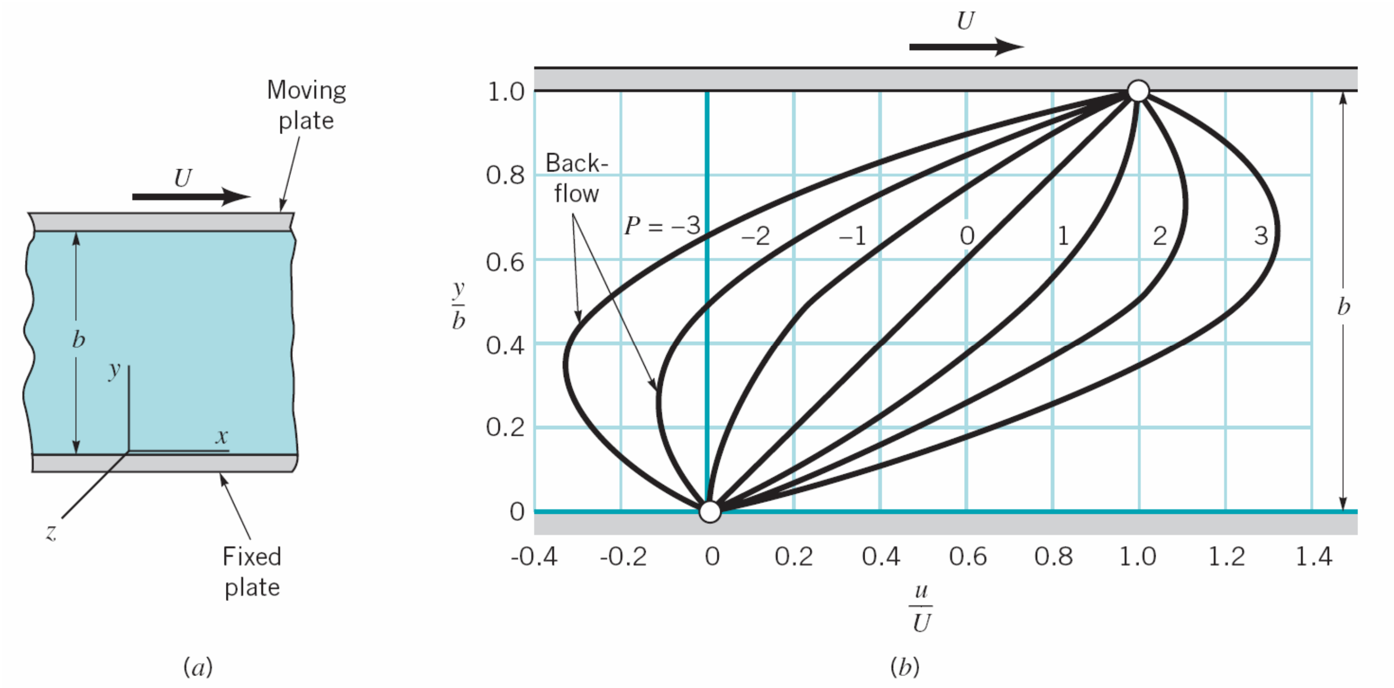
$$v = 0; w = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$$

$$\frac{du}{dy} = \frac{1}{\mu}\frac{\partial p}{\partial x}y + C_1 \Rightarrow u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + C_1 y + C_2$$

$$p = -\rho g y + f(x)$$

Boundary condition (no slip) $u(0) = 0; u(b) = U$

Please find velocity profile and flow rate

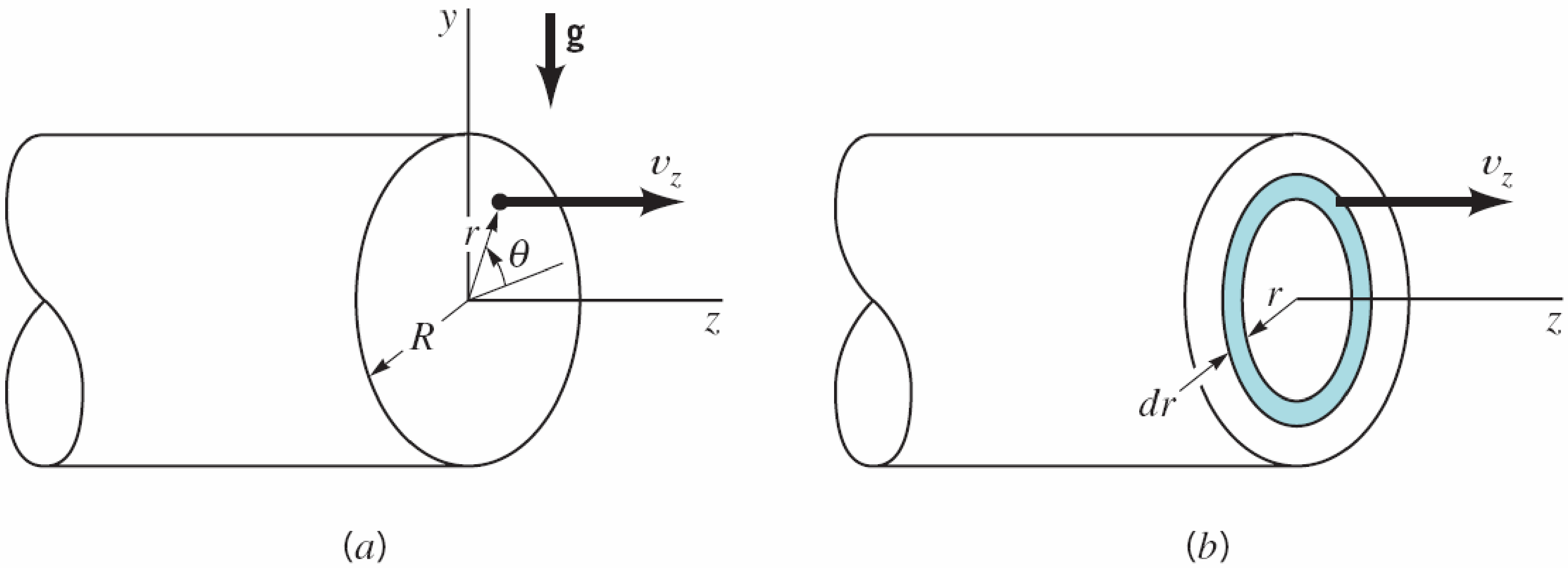


Boundary condition (no slip) $u(0) = 0; u(b) = U$

Velocity profile $u = U\dfrac{y}{b} + \dfrac{1}{2\mu}\dfrac{\partial p}{\partial x}(y^2 - by)$

$$\frac{u}{U} = \frac{y}{b} - \underbrace{\frac{b^2}{2\mu U}\frac{\partial p}{\partial x}}_{\text{Dimensionless parameter P}}\left(\frac{y}{b}\right)\left(1 - \frac{y}{b}\right)$$

Hagen-Poiseuille flow



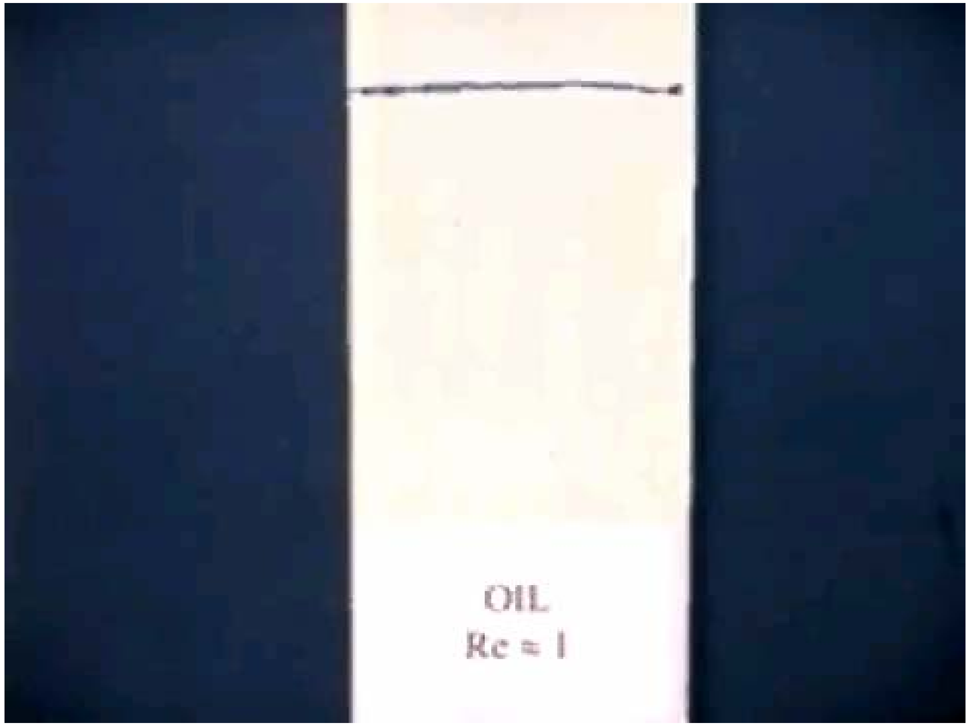
Poiseuille law:

$$Q = \frac{\pi R^4 \Delta p}{8\mu l};$$

$$v = \frac{R^2 \Delta p}{8\mu l}$$

Laminar flow

- The velocity distribution is parabolic for steady, laminar flow in circular tubes. A filament of dye is placed across a circular tube containing a very viscous liquid which is initially at rest. With the opening of a valve at the bottom of the tube the liquid starts to flow, and the parabolic velocity distribution is revealed. Although the flow is actually unsteady, it is quasi-steady since it is only slowly changing. Thus, at any instant in time the velocity distribution corresponds to the characteristic steady-flow parabolic distribution.



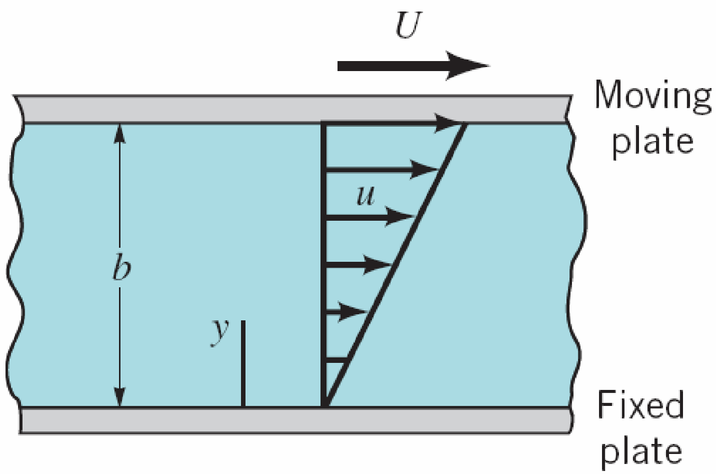
Problems

- **6.2** A certain flow field is given by equation:

$$\vec{V} = (3x^2 + 1)\vec{i} - 6xy\vec{j}$$

  Determine expression for local and convective components of the acceleration in x and y directions

- **6.8** An incompressible viscous fluid is placed between two large parallel plates. The bottom plate is fixed and the top moves with the velocity U. Determine:
  - volumetric dilation rate;
  - rotation vector;
  - vorticity;
  - rate of angular deformation.

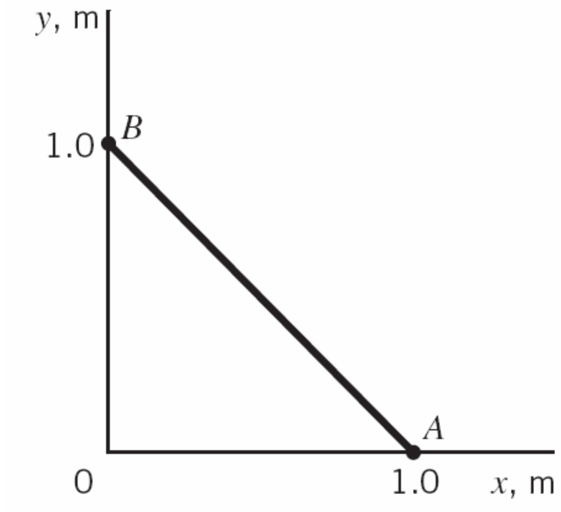


$$u = U\frac{y}{b}$$

Problems

- **6.22** The stream function for an incompressible flow

  sketch the streamline passing through the origin; determine of flow across the strait path AB



- **6.74** Oil SAE30 at 15.6C steadily flows between fixed horizontal parallel plates. The pressure drop per unit length is 20kPa/m and the distance between the plates is 4mm, the flow is laminar.
  Determine the volume rate of flow per unit width; magnitude and direction of the shearing stress on the bottom plate; velocity along the centerline of the channel