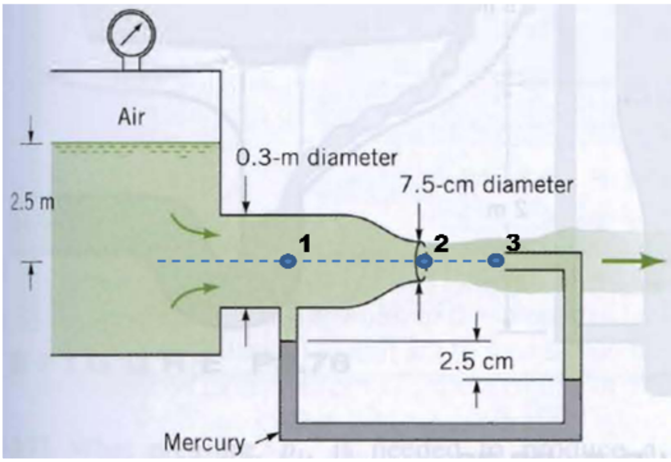
Problem 1. Water flows steadily from a large closed tank as shown in the figure. Assuming the viscous effects negligible, determine the **flow rate** and the **air pressure** in the space above the surface of water.



Solution. First we apply the Bernoulli equation between the points 1 and 3:

$$\frac{\rho v^2}{2} + P_1 = P_3$$

(the flow is stopped at the point 3 and the velocity is zero there).
The difference between P3 and P1 is read by the manometer as:
$P_3 - P_1 = 2.5cm * (\rho_{mercury} - \rho_{water})$=3.087kPa
Hence, the velocity at point 1 can be found to 2.48 m/s and the volume flow rate is:
$Q = v * \pi * (0.3/2)^2 = \mathbf{0.175\ m^3/s}$
That answers the question (a).

The question (b) is a bit trickier then it might look like, as we don't know the absolute pressure at point 1, however we can refer to the point 2 and a point on the surface of the volume. The point 2 is open to the atmosphere, the pressure there is 0. On the other hand, the velocity at the surface can be assumed 0.
The velocity at the point 2 can be found from the continuity condition:

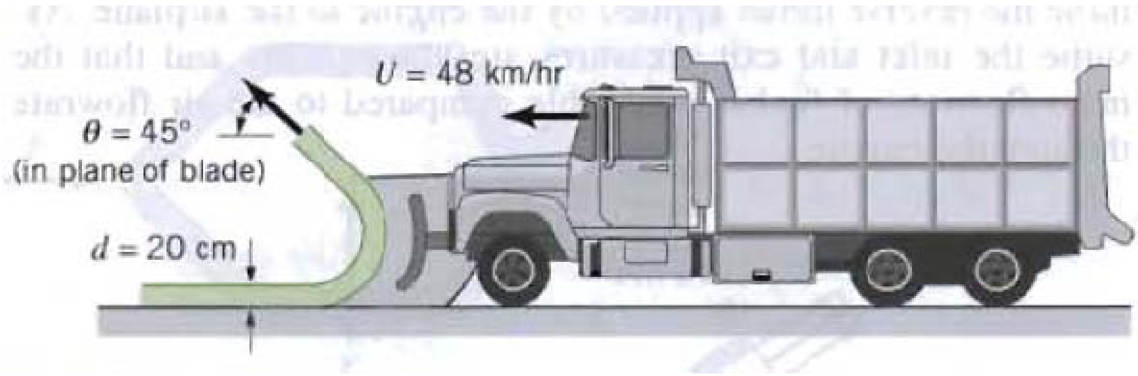
$$v2 * \pi * \left(\frac{0.075}{2}\right)^2 = v1 * \pi * \left(\frac{0.3}{2}\right)^2$$

$$\frac{\rho (v2)^2}{2} = P_{air} + \rho g * 2.5$$

The air pressure can be found as **762 kPa**.

Problem 2.

A snowplow mounted on a truck clears a path 3.5m wide through the heavy wet snow with a density of 160 kg/m^3 as shown in the figure. The truck travels at 48 km/h. Estimate the **force** required to push the snow.

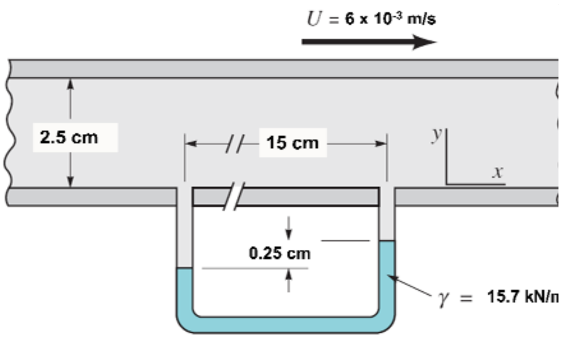


Solution. From the Reynolds transport theorem in the coordinates of the moving truck:

$$F = U * (-U) * \rho * A - U * U * \rho * A * \cos(45°)$$

Where A is the area of inlet and outlet (0.2*3.5 m^2), U – truck velocity in m/s.
Hence, the force exerted by the truck is F= **-33990 N**.

Problem 3. A viscous fluid (specific weight 1.26 kN/m^3, viscosity 1.4 Pa.s) is contained between two infinite, horizontal parallel plates as shown in the figure. The fluid moves between the plates under the action of a pressure gradient and the upper plate moves with velocity U=6x10^{-3} m/s, while the bottom plate is fixed. Calculate the **distance** from the bottom plate where velocity has a maximum. Assume laminar flow.



Solution. Solution of the Navier-Stokes equation for flow between the parallel plates looks like that:

$$u = \frac{1}{2\mu} * \frac{\partial p}{\partial x} * y^2 + c_1 y + c_2$$

Where c1 and c2 can be found from the boundary conditions v(0)=0 and v(h)=U, h is the height of the channel.

$$u = \frac{1}{2\mu} * \frac{\partial p}{\partial x} * y^2 + \left(\frac{U}{h} - \frac{1}{2\mu} * \frac{\partial p}{\partial x} * h\right) y$$

The maximum is located at $\partial u/\partial y$=0.
After some arithmetic we can find that:

$$y_{max} = \frac{h}{2} - \frac{\mu U}{h\left(\frac{\partial P}{\partial x}\right)}$$

$$\frac{\partial P}{\partial x} = -\frac{\Delta P}{l} = -\frac{0.0025 * (15.7 - 1.26) * 10^3}{0.15} = \frac{36\,Pa}{0.15\,m} = -240\frac{Pa}{m}$$

At the current values $\boldsymbol{y_{max} \approx 0.0125m}$
(approx. half height)

Problem 4. The diameter of ***d*** of the dots made by an ink jet printer depends on the ink viscosity $\boldsymbol{\mu}$, density $\boldsymbol{\rho}$, surface tension $\boldsymbol{\sigma}$, as well as jet speed ***V*** and diameter ***D***. Determine the set of **Pi-terms** to describe the process.

Solution. We have 6 variables and all 3 reference dimensions, so we choose 6-3=3 non-repeating variables and Pi-terms.

Non-repeating variables: ***d***, $\boldsymbol{\mu}$, $\boldsymbol{\rho}$

Repeating variables: $\boldsymbol{\sigma}$, ***V***, ***D***

I will choose F, T, L as few of my variables will look simpler in terms of force.

Please recall, I have to choose **d** as a non-repeating, the rest is more or less up to me.

As a result I find:

$$\Pi_1 = \frac{d}{D}$$

$$\Pi_2 = \frac{\mu v}{\sigma}$$

$$\Pi_3 = \frac{\rho v^2 D}{\sigma}$$

So,

$$\frac{\boldsymbol{d}}{\boldsymbol{D}} = \boldsymbol{\varphi}\left(\frac{\mu v}{\sigma}, \frac{\rho v^2 D}{\sigma}\right)$$

*) Please note, that Pi-terms are not unique, if you got different ones it doesn't necessarily means they are wrong!

*) If you solve the problem in MLT reference set, with the same choice of repeating and non-repeating variables you will get the same Pi-terms. Common pr oblem I find in your exam work is related to the dimensions of surface tension $\boldsymbol{\sigma}$: it's measured in N/m.
Hence in FLT system it is $F*L^{-1}$ and in MLT system it is $M*T^{-2}$.

Problem 5. A fluidic circuit in an SPR apparatus consists of several pieces of tubing connected to the rectangular microfluidic chambers (12mm long, 2mm wide, 50µm high, above the sensorchip as shown in the figure. Calculate the **force** on the 14mm diameter plunger of a syringe pump required to propel the liquid through the system at 100ul/min.

What will be the **Re number** for the flow above the sensorchip (in the rectangular microfluidic channels)?

Solution.
As everything is connected in series we can calculate pressure drop at each element and sum them together.
Flow in metric units: 1.67×10^{-9} m^3/s
In a circular channel, the pressure drop is:

$$\Delta P = \frac{128\mu l Q}{\pi D^4}$$

So, the total pressure drop in the tubing is:

$$\Delta P = \frac{128*10^{-3}*1.67*10^{-9}}{\pi D^4} * \left(\frac{1.1}{(0.5*10^{-3})^4} + \frac{0.35}{(0.125*10^{-3})^4}\right) = 98742 \text{ Pa}$$

For the chambers we can use for example the approach with hydraulic diameter and friction factors (see table in MYO, p.426):

$$\Delta P = f * \frac{l\rho v^2}{2D}; f = \frac{C}{Re_h}$$

$$D_h = \frac{2ab}{a+b} = 9.75 * 10^{-5}$$

$$Re_h = \frac{\rho v D_h}{\mu} = \mathbf{1.628}$$

$$\Delta P = 2 * \frac{96}{Re_h} * \frac{l\rho v^2}{2D} = 2024\, Pa$$

So, the total pressure drop and the force can be calculated:
$F = \pi * (7 * 10^{-3})^2 * (2024 + 98742) = \mathbf{15.5\, N}$

*) It is not necessary to use Bernoulli equation in order to find the pressure at the syringe plunger. As the flow is very slow, the pressure difference between the tubing and the syringe is negligible (<1%).

Problem 6. A voltage of 10kV is applied between the two ends of a 10cm long channel of diameter 50 µm fabricated in glass. The channel is filled with 1mM solution of NaCl.
a.) Calculate the **volumetric flow rate** in µl/min if glass has a zeta potential of -60mV at this condition.
b.) For flows through tubes in the presence of both electric field and pressure gradient, the total flow rate can be calculated by adding flow rates caused by electric field and pressure gradient. If the ends of our channel are sealed (Q_{total}=0), calculate the **pressure difference** between the ends of the tube created by the electroosmotic effect.

Solution. As the channel diameter is much larger than the Debye layer thickness, we can assume plug flow situation. Therefore:
$v_{eo} = \frac{\varepsilon\zeta}{\mu} * E = \frac{\varepsilon\zeta}{\mu} * \frac{V}{l} = 8.854 \cdot 10^{-12} * 78 * \frac{0.06}{10^{-3}} * \frac{10^4}{0.1} = 4.14 * 10^{-3} \text{m/s}$

$Q = v_{eo} * \pi * d^2/4$=8.1*10^{-12} m^3/s=**0.5 µl/min**

In the second part of the exercise we need to find pressure required to create the same volumetric flow as calculated above.

$$\Delta P = \frac{128\mu l Q}{\pi D^4} = \mathbf{84.5\ kPa}$$

*) in this situation we physically have flow in both direction present in the capillary and the pressure build up.