Lecture 3

Fluid Kinematics: Velocity field, Acceleration, Reynolds Transport Theorem and its application

Aims

- Describing fluid flow as a field
- How the flowing fluid interacts with the environment (forces and energy)

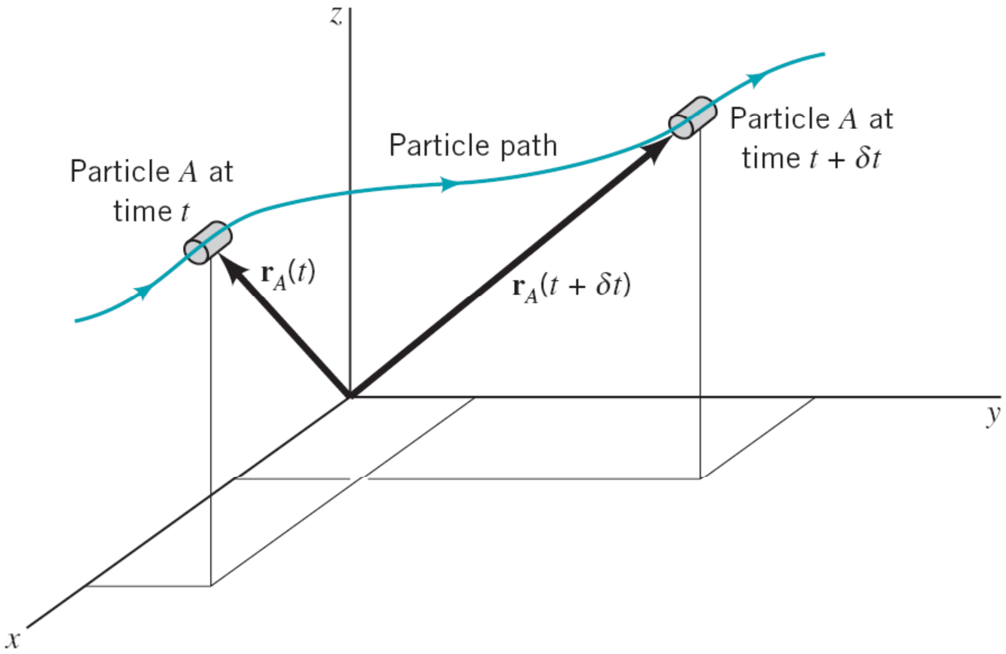
Lecture plan

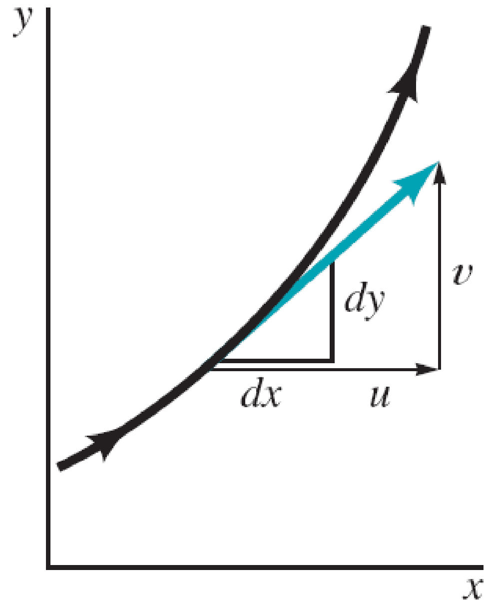
- Describing flow with the fields: Eulerian vs. Lagrangian description.
- Flow analysis: Streamlines, Streaklines, Pathlines.
- How to perform calculations in the field description: the Material Derivative
- Reynold’s Transport Theorem
- Application of Reynolds transport theorem: Continuity, Momentum and Energy conservation

Velocity field

- **field representation** of the flow: flow is represented as a function of spatial coordinates (map)
- example: velocity field

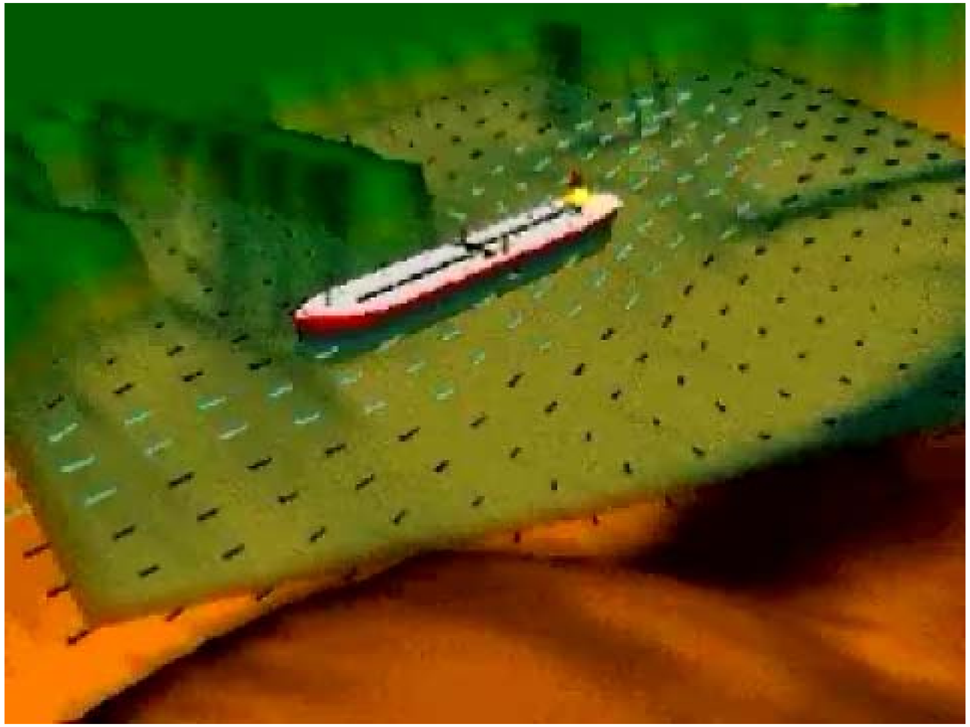
$$\vec{V} = u(x,y,z,t)\vec{i} + v(x,y,z,t)\vec{j} + w(x,y,z,t)\vec{k}$$





$$\frac{dy}{dx} = \frac{v}{u}$$

Velocity field: example



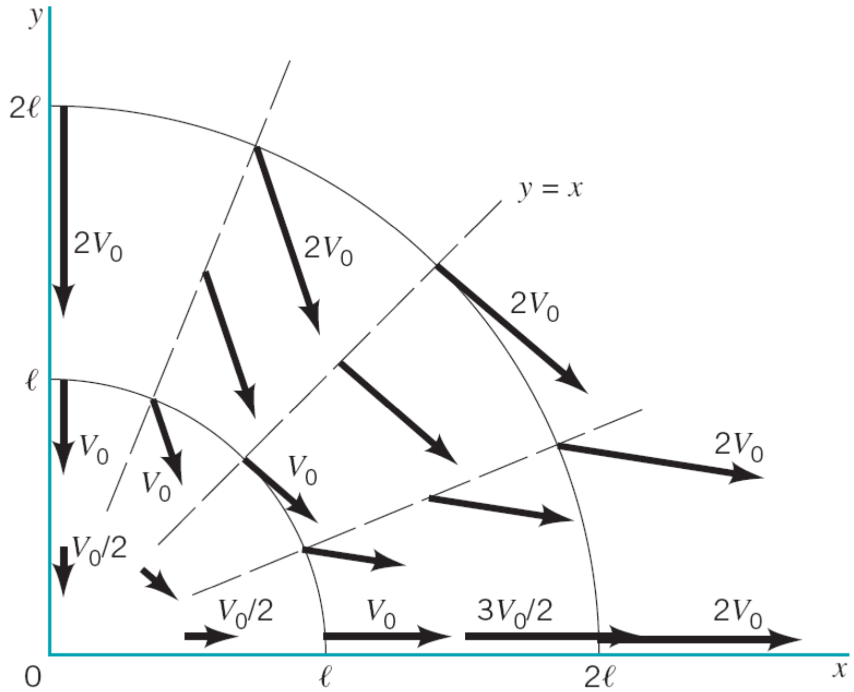
The calculated velocity field in a shipping channel is shown as the tide comes in and goes out. The fluid speed is given by the length and color of the arrows. The instantaneous flow direction is indicated by the direction that the velocity arrows point.

Velocity field representation

Velocity field is given by:

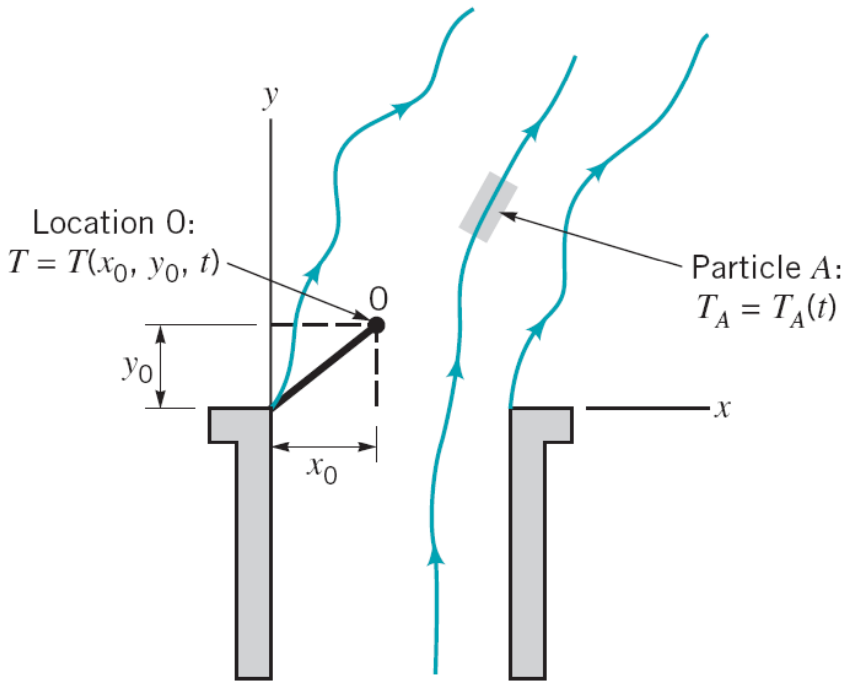
$$\vec{v} = (v_0 / l)(x\vec{i} - y\vec{j})$$

- Sketch the field in the first quadrant
- find where velocity will be equal to $\boldsymbol{v_0}$



Eulerian and Lagrangian flow description

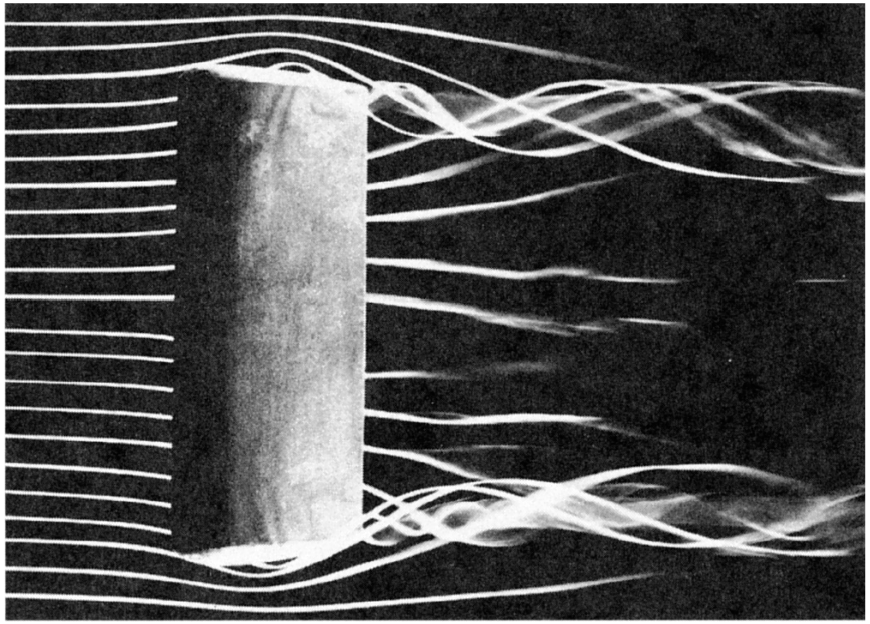
- **Eulerian method** – field concept is used, flow parameters (T, P, v etc.) are measured in every point in space vs. time
- **Lagrangian method** – an individual fluid particle is followed, parameters associated with this particle are followed in time



Example: Smoke coming out of a chimney

1D, 2D and 3D flow

- In most of flow situations the flow is three-dimensional, though in many situations it’s possible to reduce it to 2D or even 1D flow.

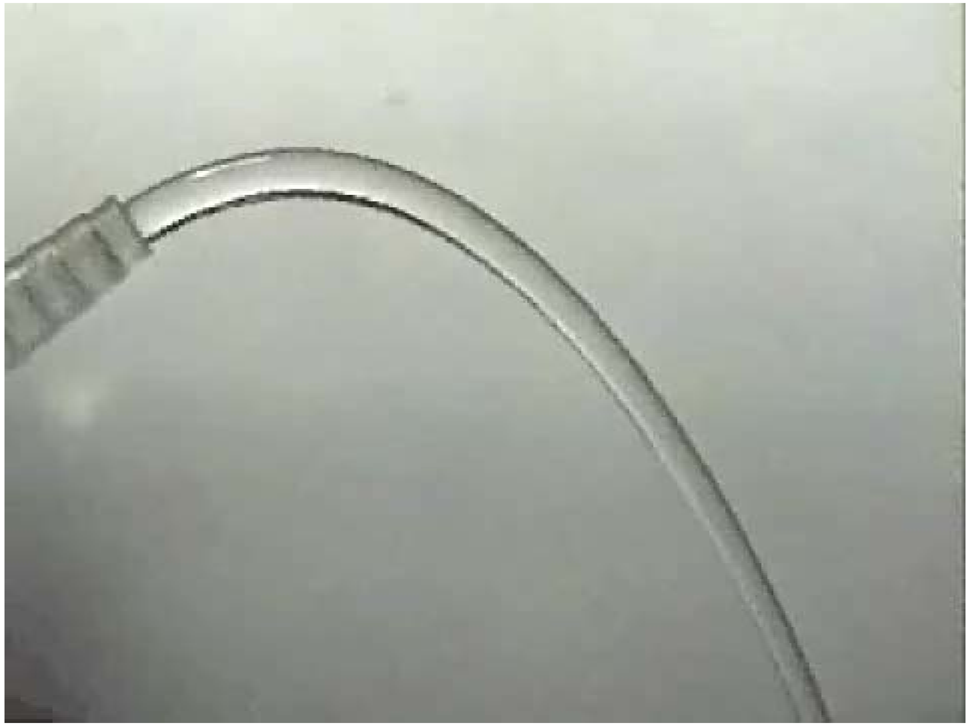


Flow visualization of the complex three-dimensional flow past a model airfoil

The flow generated by an airplane is made visible by flying a model Airbus airplane through two plumes of smoke. The complex, unsteady, three-dimensional swirling motion generated at the wing tips (called trailing vorticies) is clearly visible

Flow types

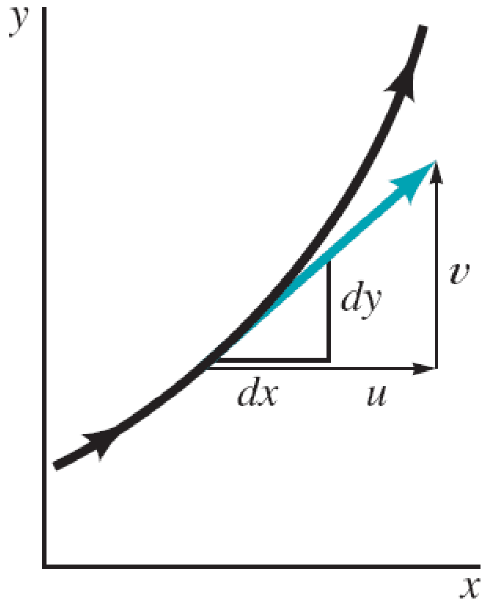
- **Steady flow** – the velocity at any given point in space doesn't vary with time. Otherwise the flow is called **unsteady**
- **Laminar flow** – fluid particles follow well defined pathlines at any moment in time, in **turbulent flow** pathlines are not defined.



Streamlines

- Streamline: line everywhere tangentional to the velocity field

$$\frac{dy}{dx} = \frac{v}{u}$$



Streamlines

Velocity field is given by:

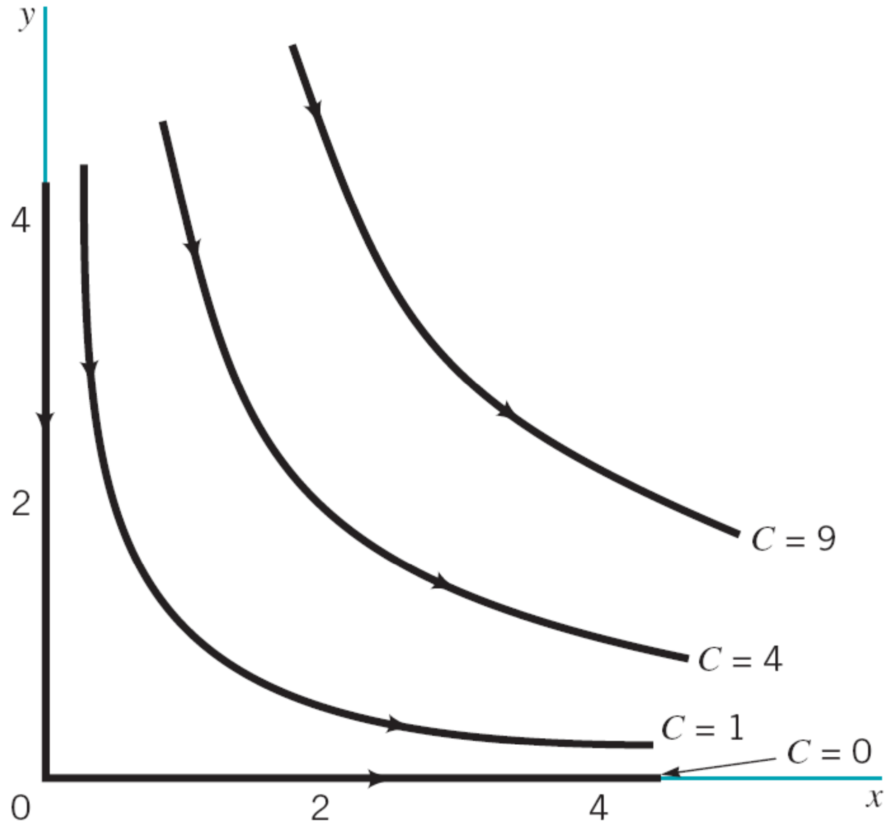
$$\vec{v} = (v_0 / l)(x\vec{i} - y\vec{j})$$

•draw the streamlines and find there equation

$$\frac{dy}{dx} = \frac{v}{u}$$

$$\frac{dy}{dx} = \frac{v}{u} = -\frac{y}{x};$$

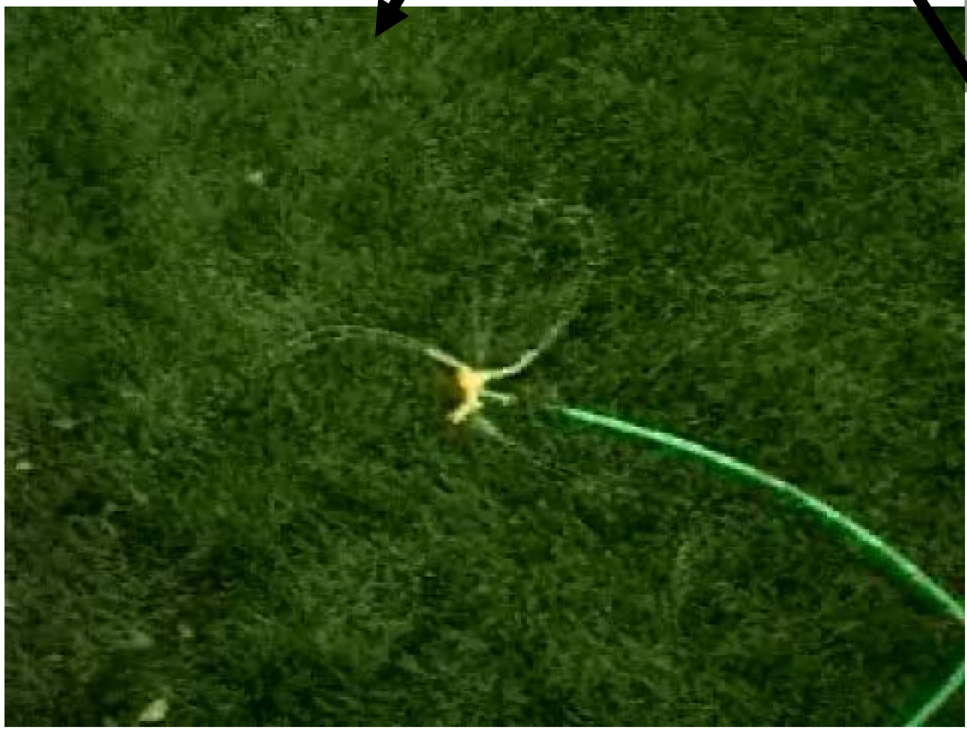
$$\ln y = \ln x + C; \; y = C' / x$$

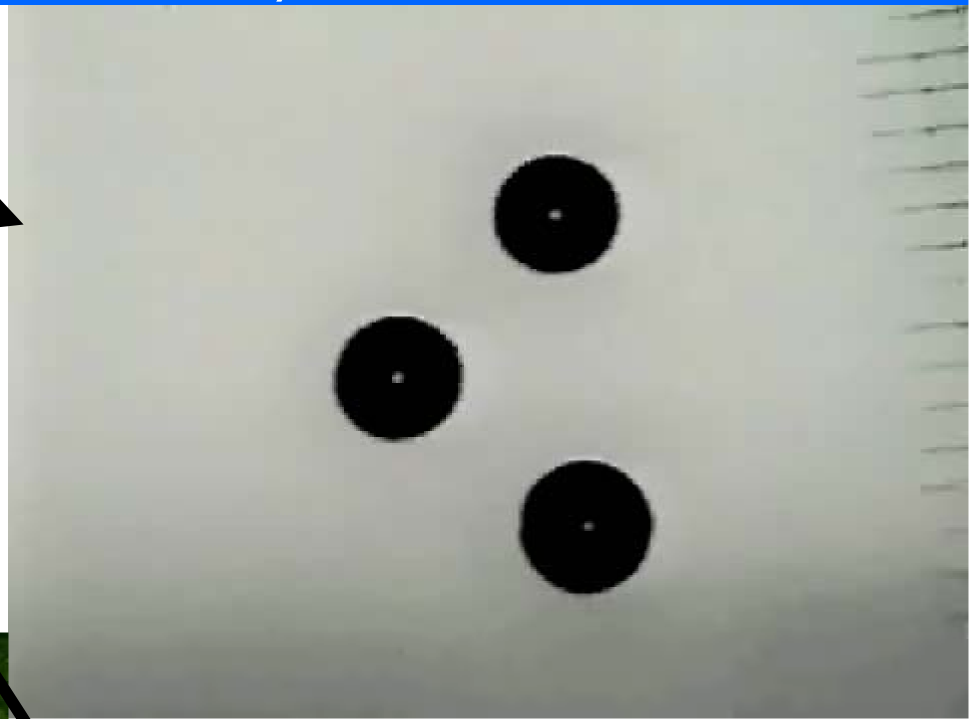


$$xy = Const$$

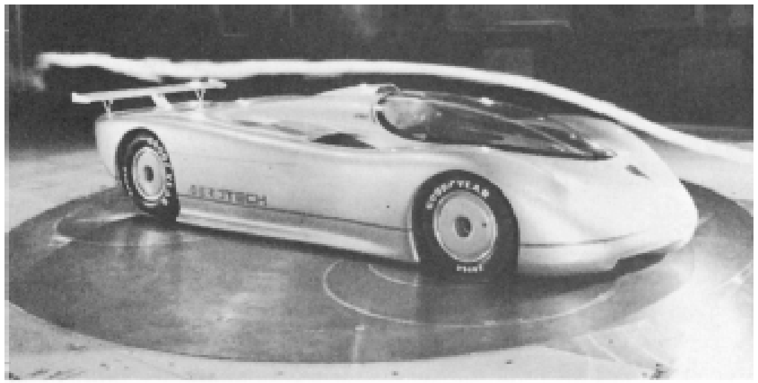
Streamlines, Streaklines, Pathlines

- Streamline – line that everywhere tangent to velocity field
- Streakline – all particles that passed through a common point
- Pathline – line traced by a given particle as it flows



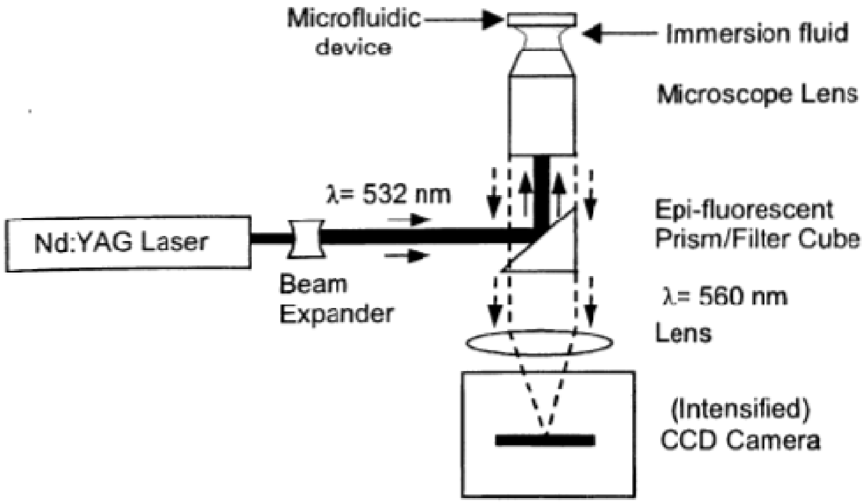


streamlines



Micro Particle Image Velocimetry

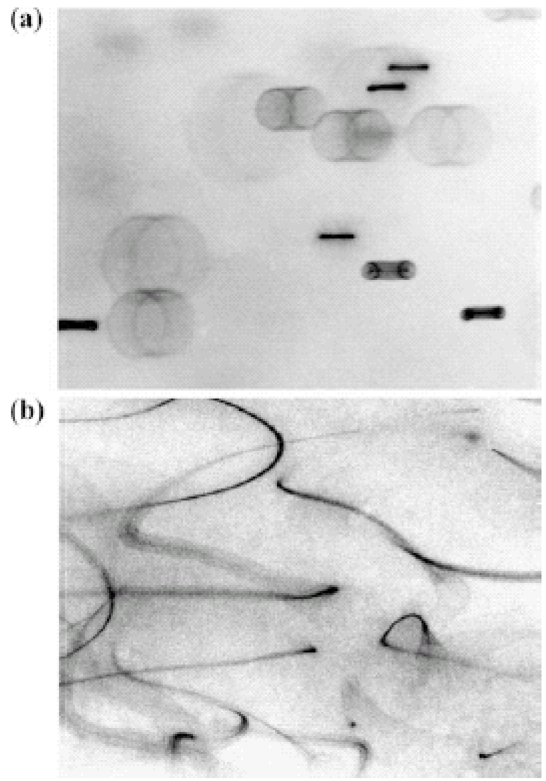
- Particle Image Velocimetry (PIV):
  - flow is seeded by small particles,
  - consecutive photographs of particles distribution are made
  - Images are sectioned into ***interrogation regions***
  - Motion of particles within interrogation region determined by image cross-correlation



Flow characterization on a microscale

– Particle Streak Velocimetry

– Particle Image Velocimetry (PIV)



Advantages:

– Full-field technique

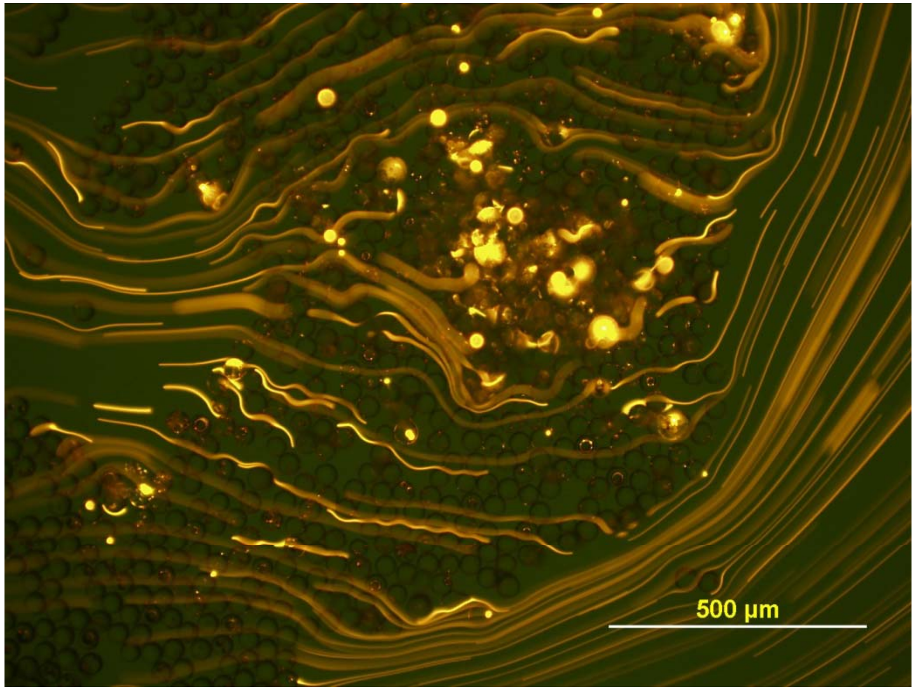
– high spacial resolution

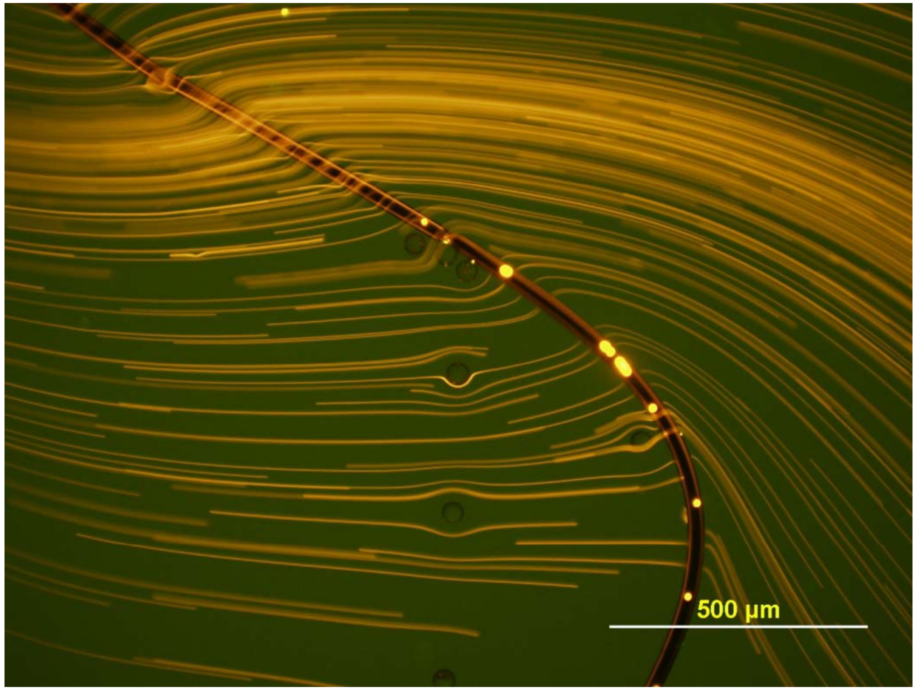
– large range of velocities covered (up to 8m/s)

– simplicity

Imaging flow in a microfluidic channel

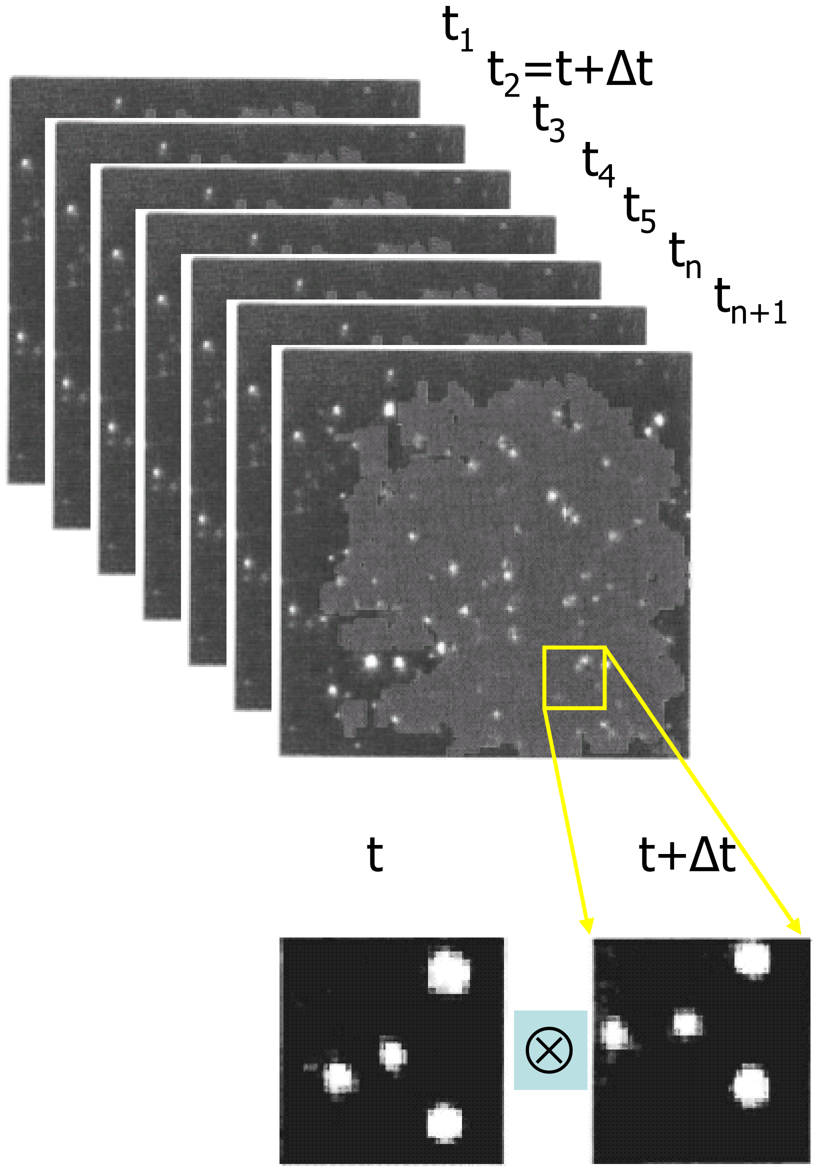
- Flow is seeded with fluorescent particles and imaged…
  (Project 5th semester Fall 2006)





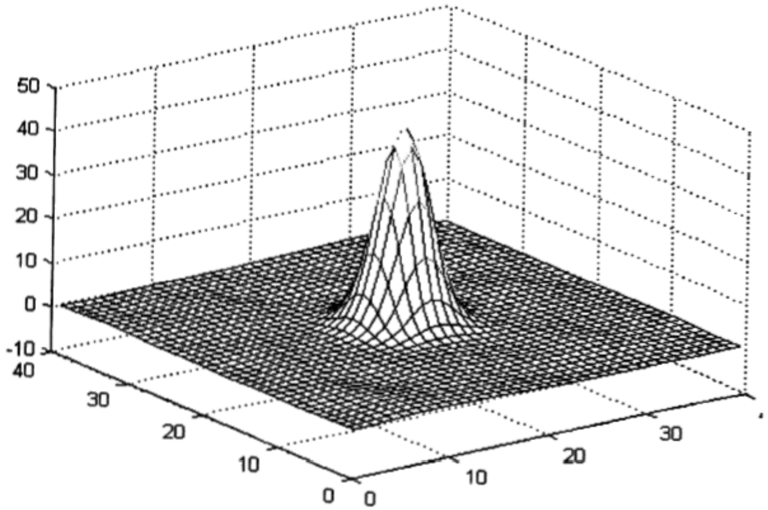
Flow through a loosely packed microspheres bed

Flow at a channel turn.
Flow is disturbed by a microwire



Cross-correlation of 2 images is calculated

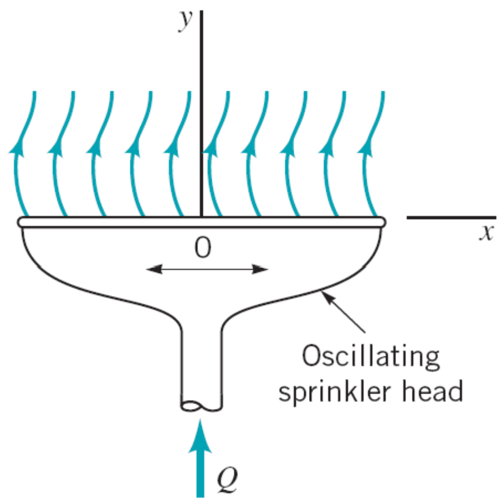
$$\Phi(m,n) = \sum_{j=1}^{q} \sum_{i=1}^{p} f(i,j) \cdot g(i+m, j+n)$$



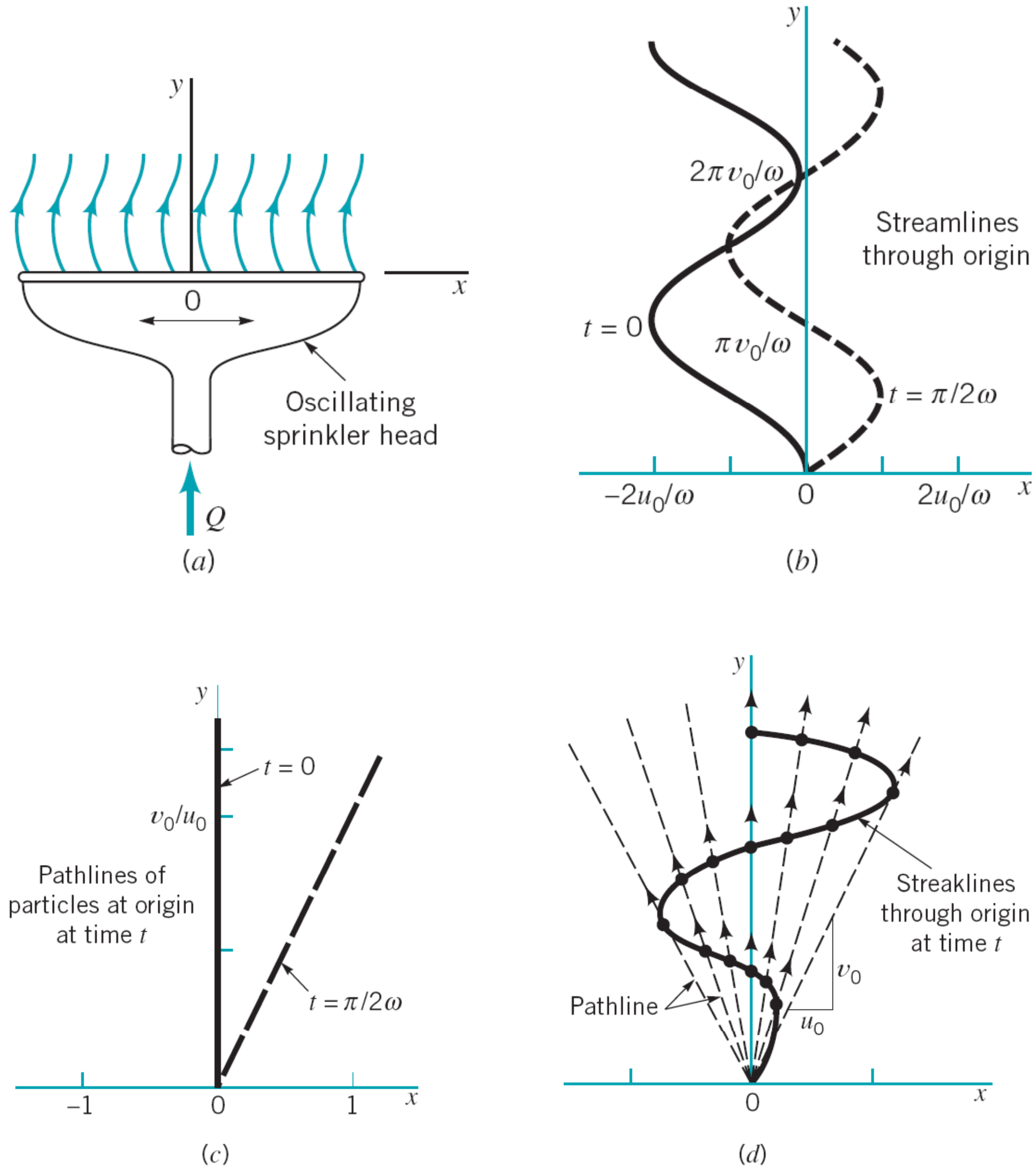
Example

Water flowing from an oscillating slit:

$$\vec{v} = u_0 \sin(\omega(t - y / v_0))\vec{i} + v_0 \vec{j}$$



(a)



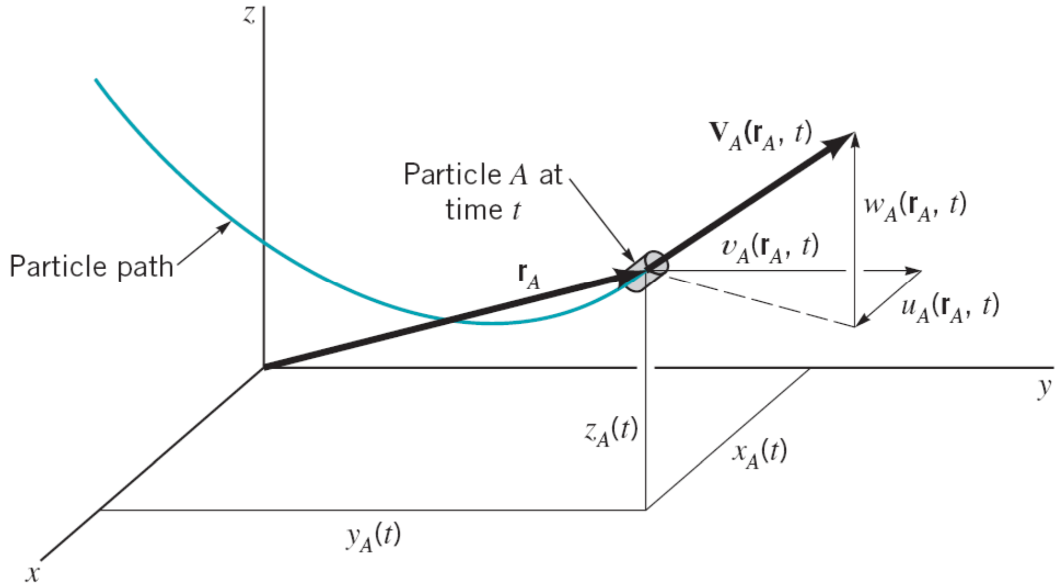
y
x
0
Oscillating sprinkler head
Q
(a)
y
$2\pi v_0/\omega$
Streamlines through origin
$t = 0$
$\pi v_0/\omega$
$t = \pi/2\omega$
$-2u_0/\omega$
0
$2u_0/\omega$
x
(b)
y
$t = 0$
v_0/u_0
Pathlines of particles at origin at time t
$t = \pi/2\omega$
-1
0
1
x
(c)
y
Streaklines through origin at time t
v_0
Pathline
u_0
0
x
(d)

Material derivative

$$V_A(r_A,t)$$

- Particle velocity

- Particle acceleration



$$a_A(r_A,t) = \frac{dV_A}{dt} = \frac{\partial V_A}{\partial t} + \frac{\partial V_A}{\partial x}\frac{\partial x_A}{\partial t} + \frac{\partial V_A}{\partial y}\frac{\partial y_A}{\partial t} + \frac{\partial V_A}{\partial z}\frac{\partial z_A}{\partial t}$$

$$\mathbf{a}(r,t) = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = \frac{D\mathbf{V}}{Dt}$$

- Material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + (\mathbf{V} \bullet \nabla)$$

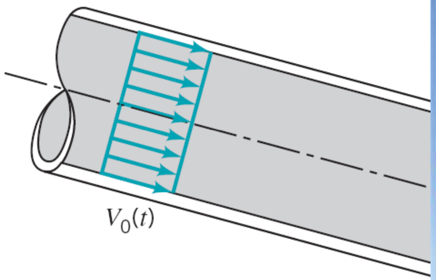
Material derivative

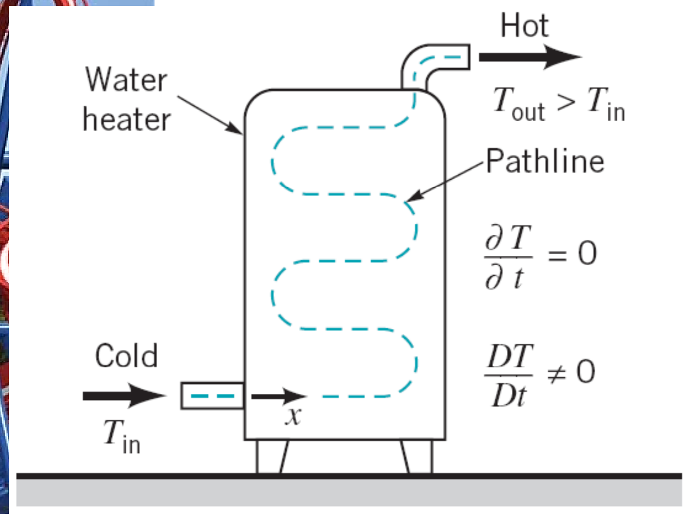
- is the rate of changes for a given variable with time for a given particle of fluid.

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + (\mathbf{V} \bullet \nabla)$$

Unsteadiness of the flow
(local acceleration)

Convective effect
(convective acceleration)





Example: acceleration

Velocity field is given by: $\vec{v} = (v_0 / l)(x\vec{i} - y\vec{j})$

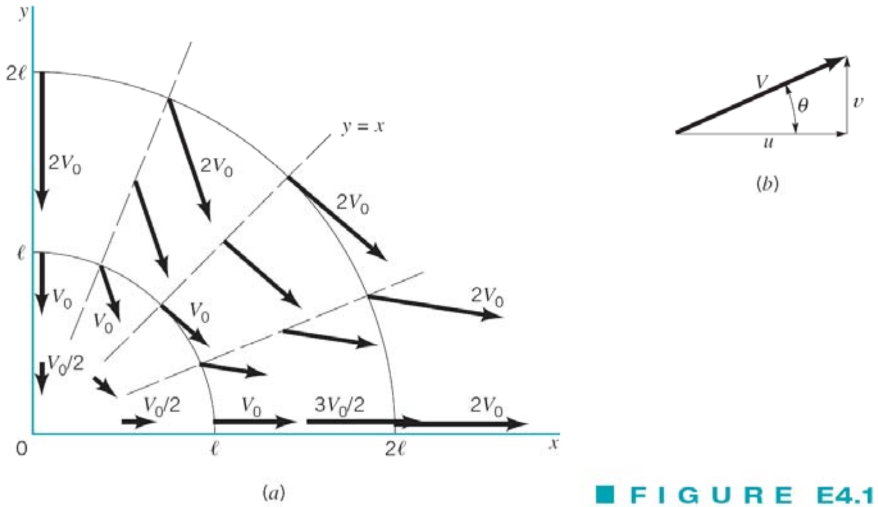
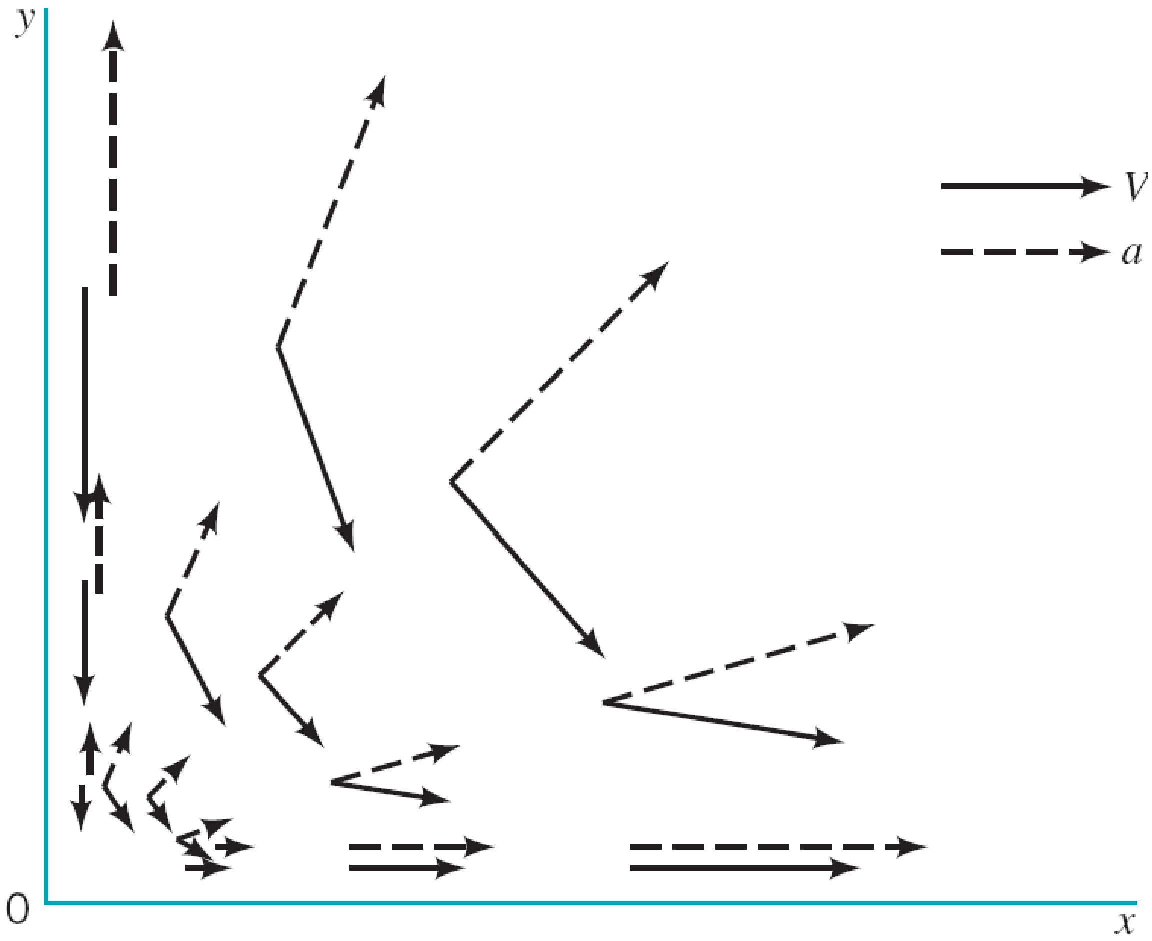


FIGURE E4.1

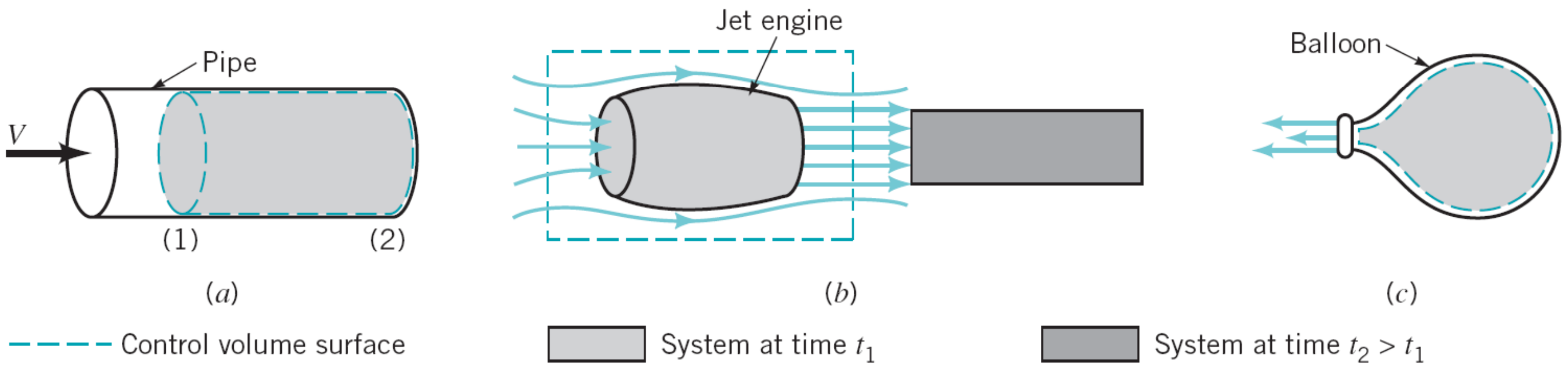
Find the acceleration and draw it on scheme



y
V
a
0
x

Control volume and system representation

- **System** – specific identifiable quantity of matter, that might interact with the surrounding but always contains the same mass
- **Control volume** – geometrical entity, a volume in space through which fluid may flow



Governing laws of fluid motion are stated in **terms of the system**, but control volume approach is essential for practical applications

Control volume: example

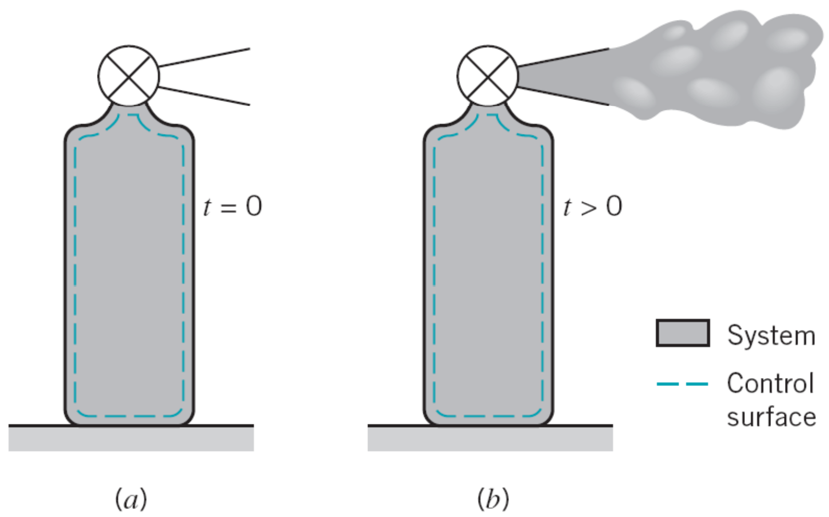
- Extensive property: B=mb (e.g. m (b=1), mv (b=v), $mv^2/2$ etc)

intensive property

$$B_{sys} = \int_{sys} \rho b d\mathcal{V}$$

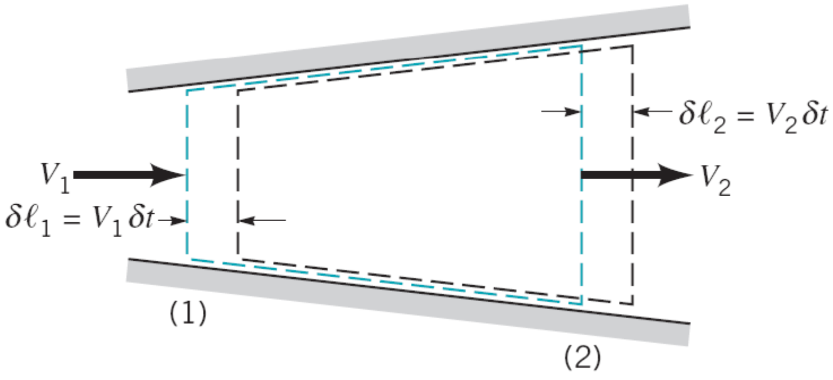
$$\frac{\partial B_{sys}}{\partial t} = \frac{\partial}{\partial t}\int_{sys} \rho b d\mathcal{V}$$

$$\frac{\partial B_{cv}}{\partial t} = \frac{\partial}{\partial t}\int_{cv} \rho b d\mathcal{V}$$

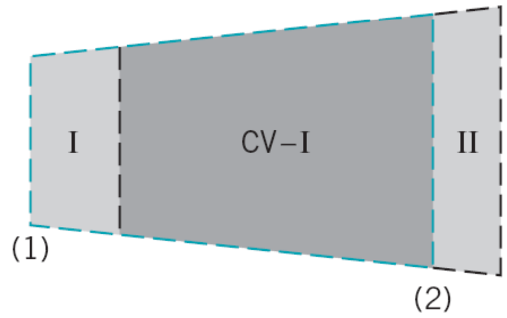


$$\frac{\partial B_{sys}}{\partial t} = \frac{\partial}{\partial t}\int_{sys} \rho d\mathcal{V} = 0; \frac{\partial B_{cv}}{\partial t} = \frac{\partial}{\partial t}\int_{cv} \rho d\mathcal{V} < 0$$

The Reynolds Transport theorem (simplified)



(a)



$t : SYS = CV$

$t+\delta t : SYS = (CV - I) + II$

(b)

Let's consider an extensive property B:

initially, at time t: $B_{sys}(t) = B_{cv}(t)$

at time t+δt: $B_{sys}(t+\delta t) = B_{cv}(t+\delta t) - B_I(t+\delta t) + B_{II}(t+\delta t)$

$$\frac{B_{sys}(t+\delta t) - B_{sys}(t)}{\delta t} = \frac{B_{cv}(t+\delta t) - B_{cv}(t)}{\delta t} - \frac{B_I(t+\delta t)}{\delta t} + \frac{B_{II}(t+\delta t)}{\delta t}$$

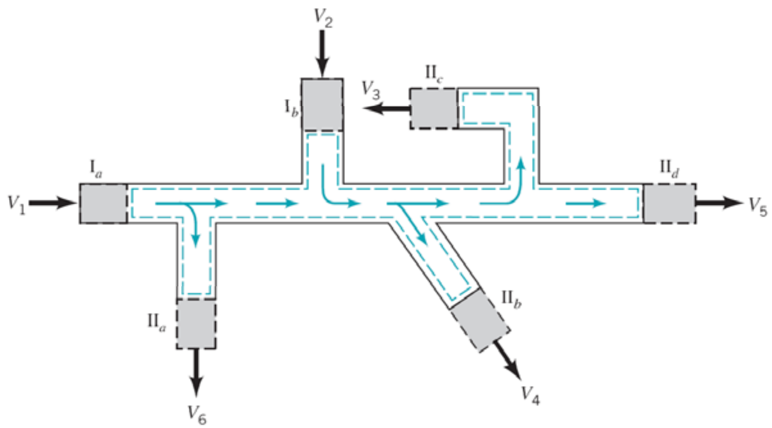
$\frac{\partial B_{cv}}{\partial t}$ inflow outflow

$$\frac{DB_{sys}}{Dt} = \frac{\partial B_{cv}}{\partial t} - \dot{B}_{in} + \dot{B}_{out}$$

- For fixed control volume with one inlet, one outlet, velocity normal to inlet/outlet

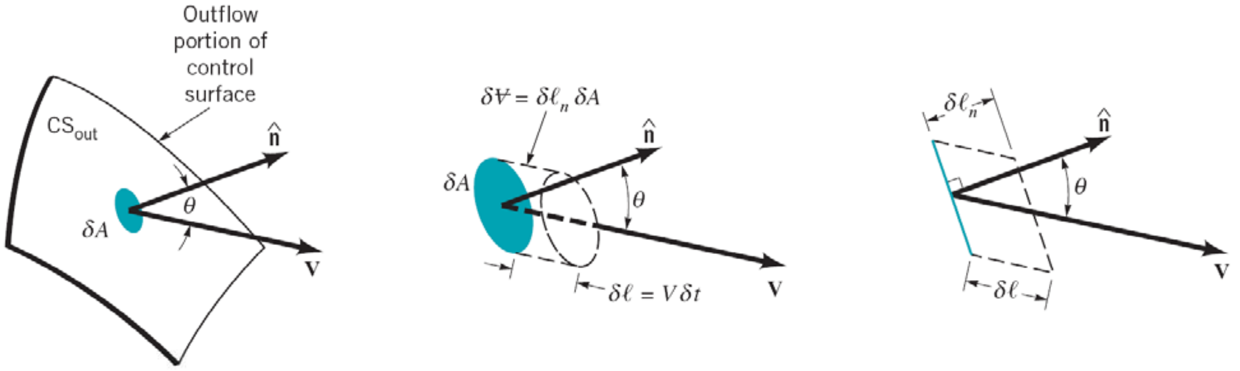
$$\frac{DB_{sys}}{Dt} = \frac{\partial B_{cv}}{\partial t} - \rho_1 A_1 V_1 b_1 + \rho_2 A_2 V_2 b_2$$

- Can be easily generalized:



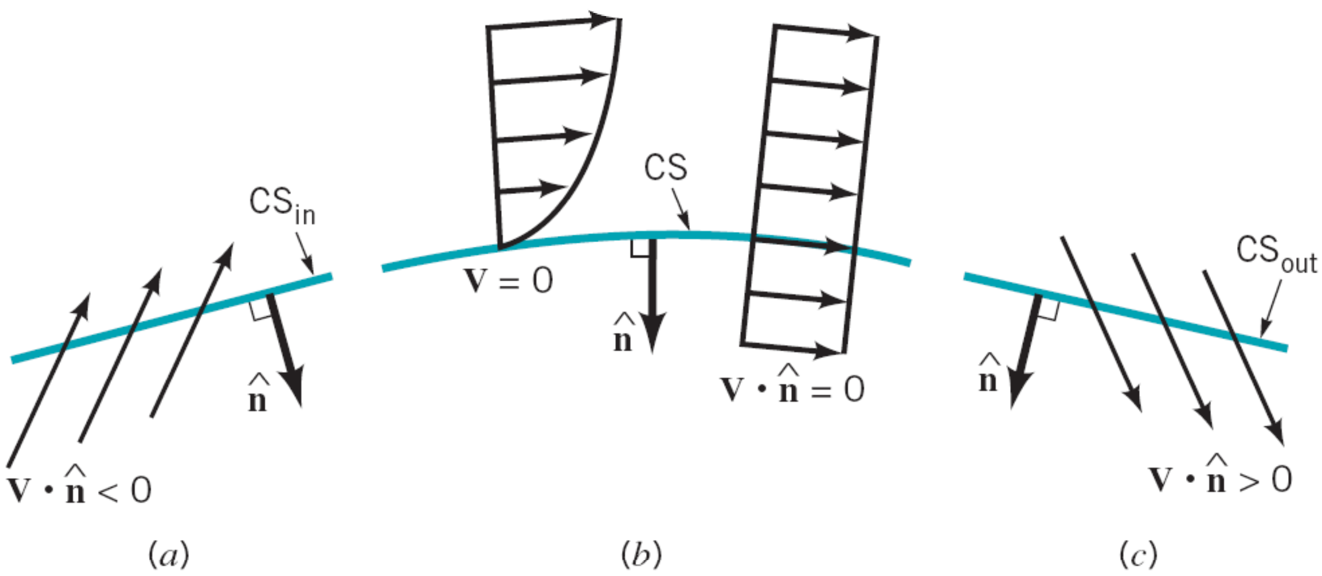
The Reynolds Transport theorem (for fixed nondeforming volume)

$$\delta B = b\rho\,\delta \forall = b\rho(V\cos\theta\,\delta t)\,\delta A$$



(a) (b) (c)

$$\dot{B}_{out} = \int_{CV_{out}} \rho b \mathbf{V}\cdot\hat{\mathbf{n}}\,dA$$



$$\frac{DB_{Sys}}{Dt} = \frac{\partial B_{CV}}{\partial t} + \int_{CV} \rho b \mathbf{V} \cdot \hat{\mathbf{n}} dA$$

General Reynolds transport theorem for fixed control volume

Application of Reynolds Transport Theorem

$$\frac{DB_{Sys}}{Dt} = \frac{\partial B_{CV}}{\partial t} + \int_{CV} \rho b \mathbf{V} \cdot \widehat{\mathbf{n}} dA$$

- We will apply it now to various properties:
  - mass (continuity equation)
  - momentum (Newton 2nd law)
  - energy

Conservation of mass

- The amount of mass in the system should be conserved:

$$\frac{DM_{Sys}}{Dt} = \frac{D}{Dt}\int_{sys} \rho dV = \frac{\partial}{\partial t}\int_{CV} \rho dV + \int_{CV} \rho \mathbf{V} \cdot \widehat{\mathbf{n}} dA = 0$$

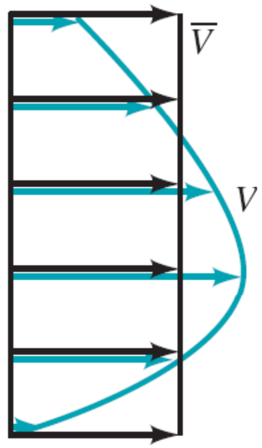
Continuity equation

Mass flow rate through a section of control surface having area A:

$$\dot{m} = \rho Q = \rho A \overline{V}$$

Volume flow rate

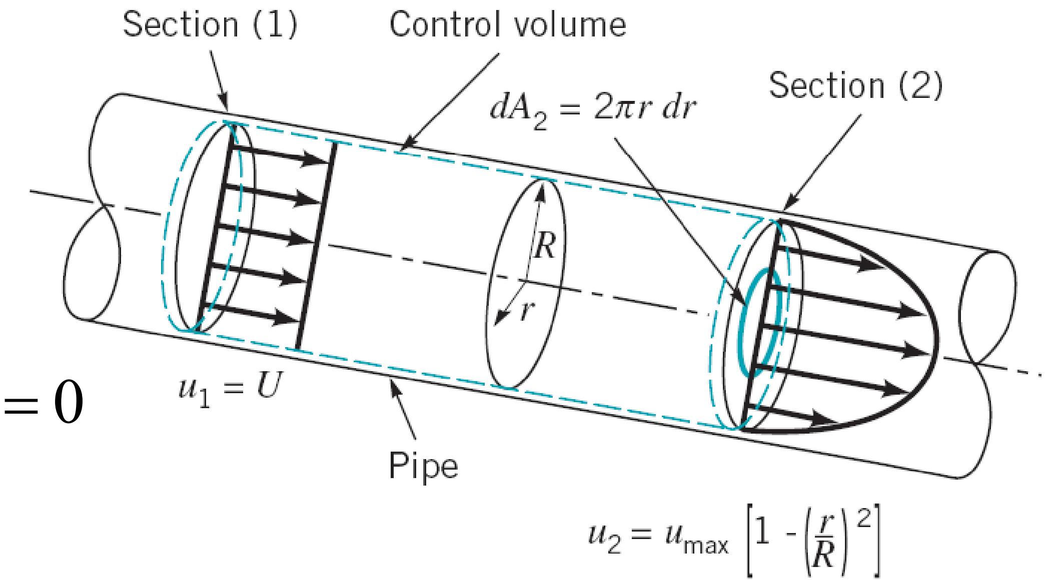
Average velocity: $\overline{V} = \dfrac{\int_A \rho \vec{V} \cdot \vec{n}\, dA}{\rho A}$



- For incompressible flow, the volume flowrate into a control volume equals the volume flowrate out of it.
- The overflow drain holes in a sink must be large enough to accommodate the flowrate from the faucet if the drain hole at the bottom of the sink is closed. Since the elevation head for the flow through the overflow drain is not large, the velocity there is relatively small. Thus, the area of the overflow drain holes must be larger than the faucet outlet area

Example

Incompressible laminar flow develops in a straight pipe of radius R. At section 1 velocity profile is uniform, at section 2 profile is axisymmetric and parabolic with maximum value u_{max}. Find relation between U and u_{max}, what is average velocity at section (2)?



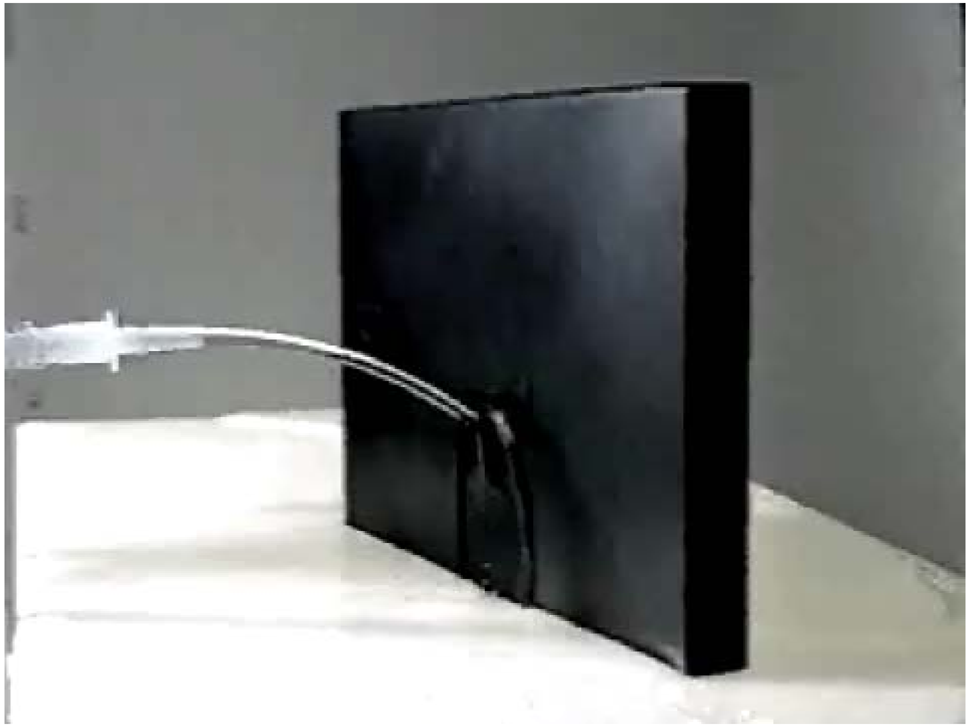
$$-\rho_1 A_1 U + \int_{A_2} \rho \vec{V} \cdot \vec{n}\, dA_2 = 0$$

$$-A_1 U + u_{max} \int_0^R \left[1 - \left(\frac{r}{R}\right)^2\right] 2\pi r dr = 0$$

$$u_{max} = 2U$$

Newton second law and conservation of momentum & momentum-of-momentum

A jet of fluid deflected by an object puts a force on the object. This force is the result of the change of momentum of the fluid and can happen even though the speed (magnitude of velocity) remains constant.



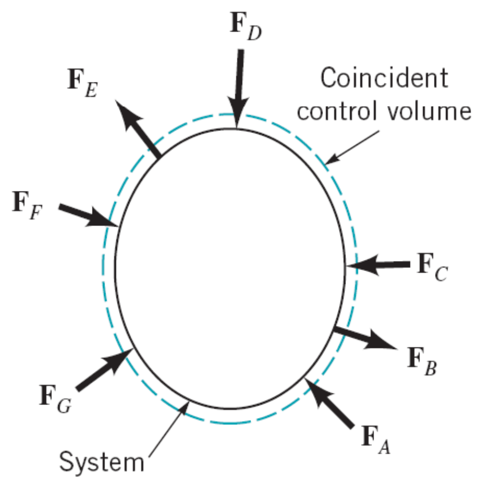
Newton second law and conservation of momentum & momentum-of-momentum

In an inertial coordinate system:

$$\frac{D}{Dt}\int_{sys} \mathbf{V}\rho dV = \sum F_{sys}$$

Rate of change of the momentum of the system

Sum of all external forces acting on the system



At a moment when system coincide with control volume:

$$\sum F_{sys} = \sum F_{\substack{\text{contents of} \\ \text{the control volume}}}$$

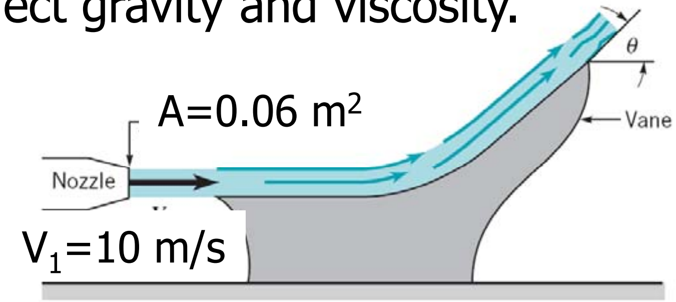
On the other hand: $$\frac{D}{Dt}\int_{sys} \mathbf{V}\rho dV = \frac{\partial}{\partial t}\int_{CV} \mathbf{V}\rho dV + \int_{CS} \mathbf{V}\rho \mathbf{V}\cdot\hat{\mathbf{n}} dA$$

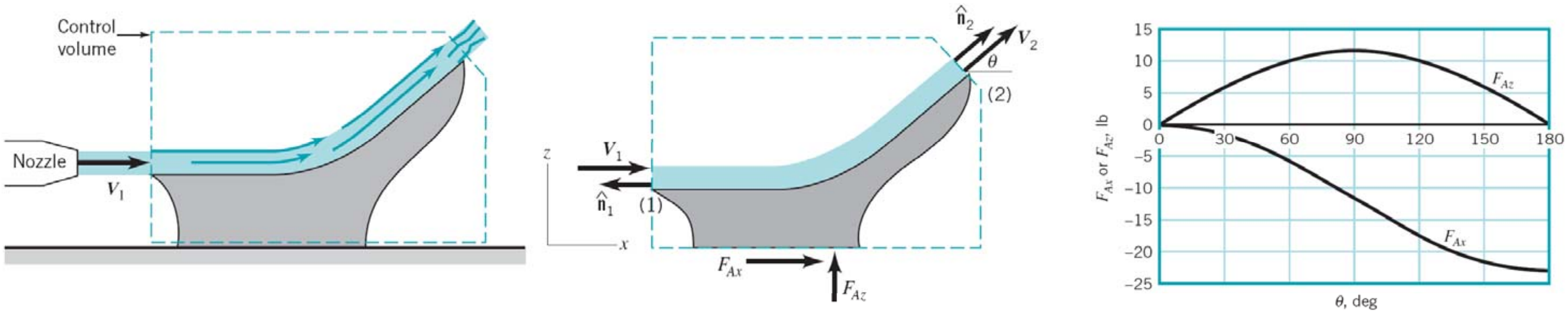
$$\frac{\partial}{\partial t}\int_{CV} \mathbf{V}\rho dV + \int_{CS} \mathbf{V}\rho \mathbf{V}\cdot\hat{\mathbf{n}} dA = \sum F_{\substack{\text{contents of} \\ \text{the control volume}}}$$

Example: Linear momentum

Determine anchoring forces required to keep the vane stationary vs angle θ.

Neglect gravity and viscosity.





$$\cancel{\frac{\partial}{\partial t}\int_{CV} u\rho d\forall} + \int_{CS} \rho u \mathbf{V}\cdot\hat{\mathbf{n}}dA = \sum F_x$$

$$\cancel{\frac{\partial}{\partial t}\int_{CV} w\rho d\forall} + \int_{CS} \rho w \mathbf{V}\cdot\hat{\mathbf{n}}dA = \sum F_y$$

$$V_1\rho(-V_1)A_1 + V_1\cos\theta\rho(V_1)A_2 = F_{Ax}$$

$$0\rho(-V_1)A_1 + V_1\sin\theta\rho(V_1)A_2 = F_{Az}$$

$$F_{Ax} = -V_1^2\rho A_1(1-\cos\theta)$$

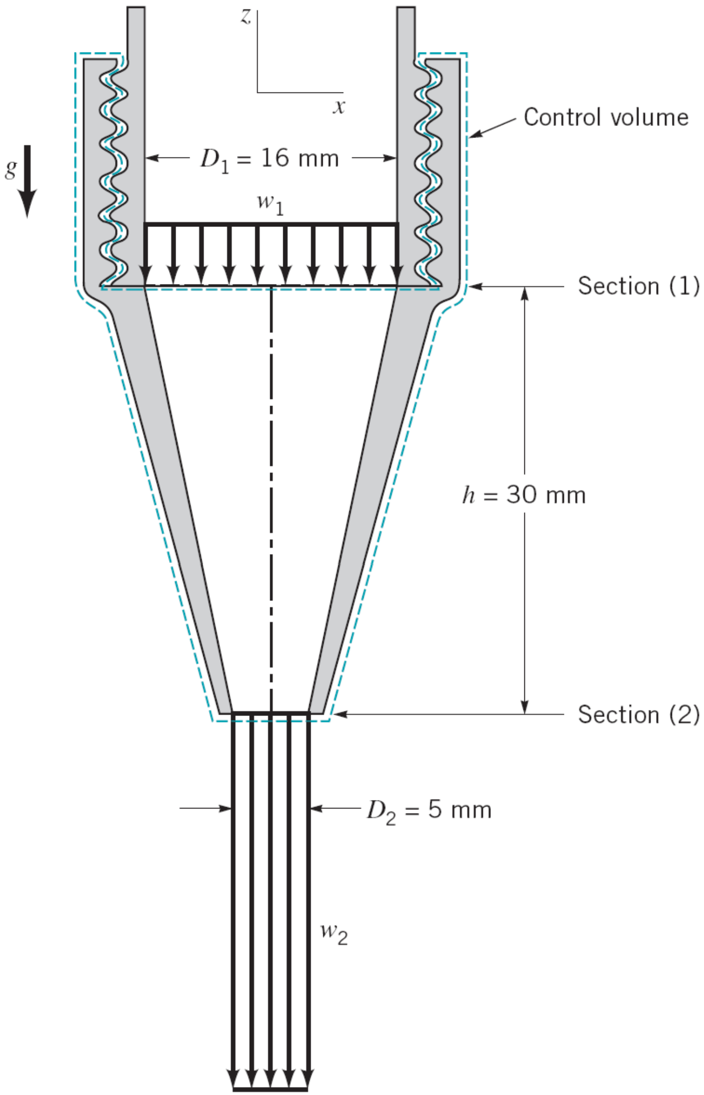
$$F_{Az} = V_1^2\rho A_1\sin\theta$$

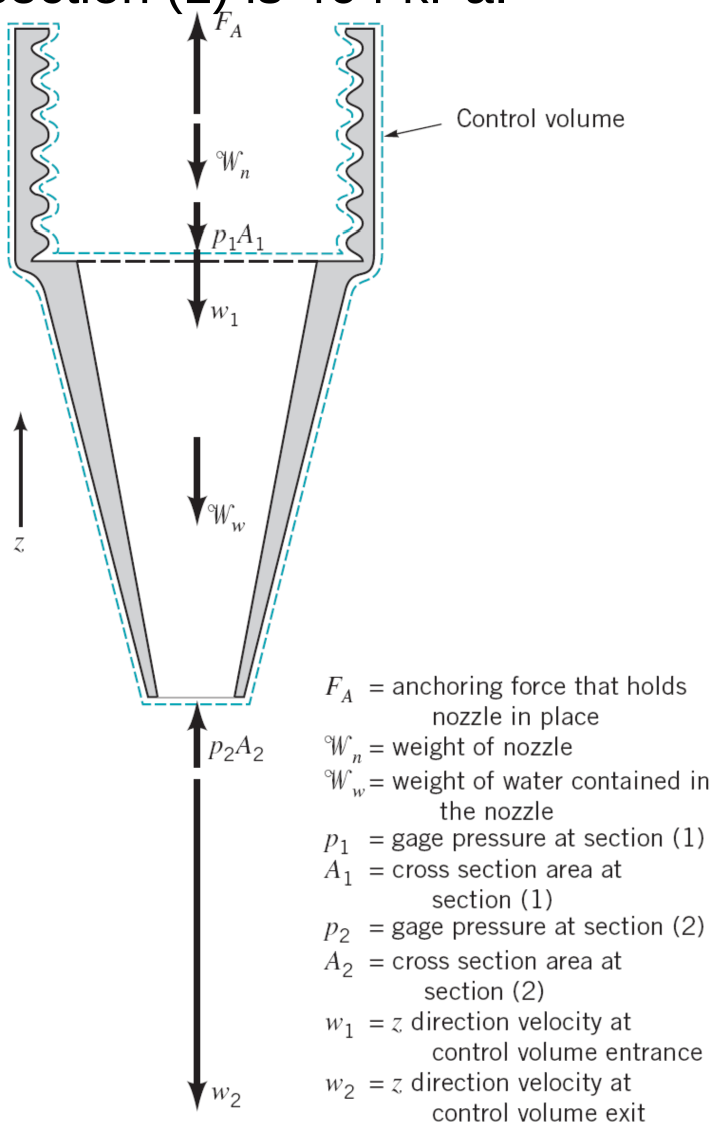
Linear momentum: comments

- Linear momentum is a vector
- As normal vector points outwards, momentum flow inside a CV involves negative **V·n** product and moment flow outside of a CV involves a positive **V·n** product.
- The time rate of change of the linear momentum of the contents of a nondeforming CV is zero for steady flow
- Forces due to atmospheric pressure on the CV may need to be considered

Example: Linear momentum – taking into account weight, pressure and change in speed

- Determine the anchoring force required to hold in place a conical nozzle attached to the end of the laboratorial sink facet. The water flow rate is 0.6 l/s, nozzle mass 0.1kg. The pressure at the section (1) is 464 kPa.

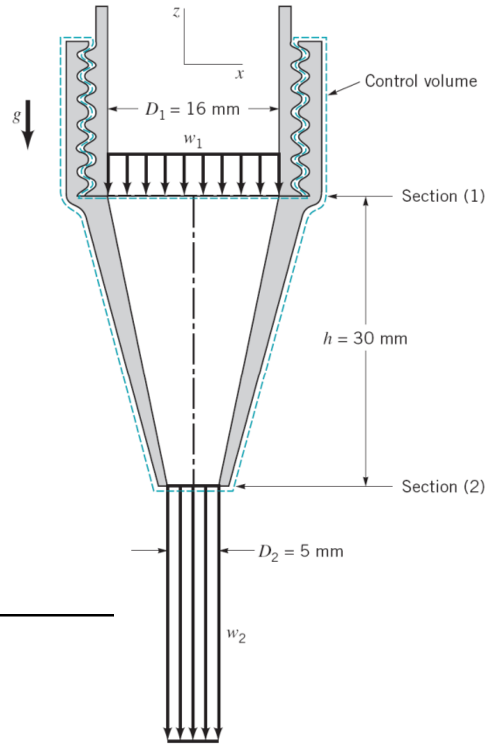


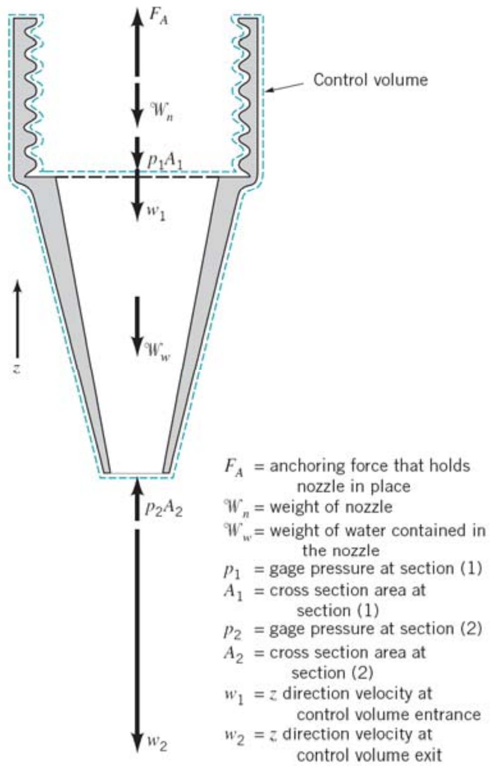


Example: Linear momentum – taking into account weight, pressure and change in speed

$$\frac{\partial}{\partial t}\int_{CV} w\rho\, d\forall + \int_{CS} w\rho\,\mathbf{V}\cdot\hat{\mathbf{n}}\, dA =$$

$$= F_A - W_n - p_1A_1 - W_w + p_2A_2$$

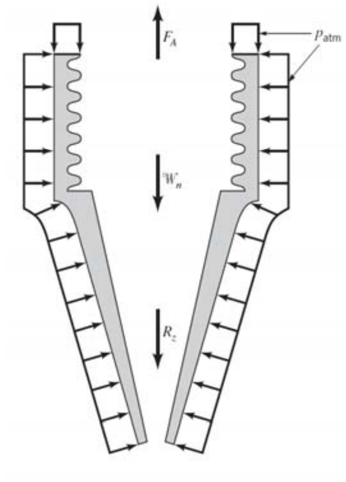




volume of a truncated cone:

$$\forall_w = \frac{1}{12}\pi h\left(D_1^2 + D_2^2 + D_1D_2\right)$$

pressure distribution:



(c)

Moment-of-Momentum Equation

The net rate of flow of moment-of-momentum through a control surface equals the net torque acting on the contents of the control volume.

Water enters the rotating arm of a lawn sprinkler along the axis of rotation with no angular momentum about the axis. Thus, with negligible frictional torque on the rotating arm, the absolute velocity of the water exiting at the end of the arm must be in the radial direction (i.e., with zero angular momentum also). Since the sprinkler arms are angled "backwards", the arms must therefore rotate so that the circumferential velocity of the exit nozzle (radius times angular velocity) equals the oppositely directed circumferential water velocity.

The Energy Equation

- The First Law of thermodynamics

$$\frac{D}{Dt}\int_{sys} e\rho dV = (\dot{Q} + \dot{W})_{sys}$$

Rate of increase of the total stored energy of the system

Net rate of energy addition by heat transfer into the system

Net rate of energy addition by work transfer into the system

Total stored energy per unit mass: $e = \widehat{u} + \frac{V^2}{2} + gz$

$$\frac{D}{Dt}\int_{sys} e\rho dV = \frac{\partial}{\partial t}\int_{CV} e\rho dV + \int_{CS} e\rho \mathbf{V}\cdot\hat{\mathbf{n}} dA$$

Rate of increase of the total stored energy of the system

rate of increase of the total stored energy of the control volume

Net rate of energy flow out of the control volume

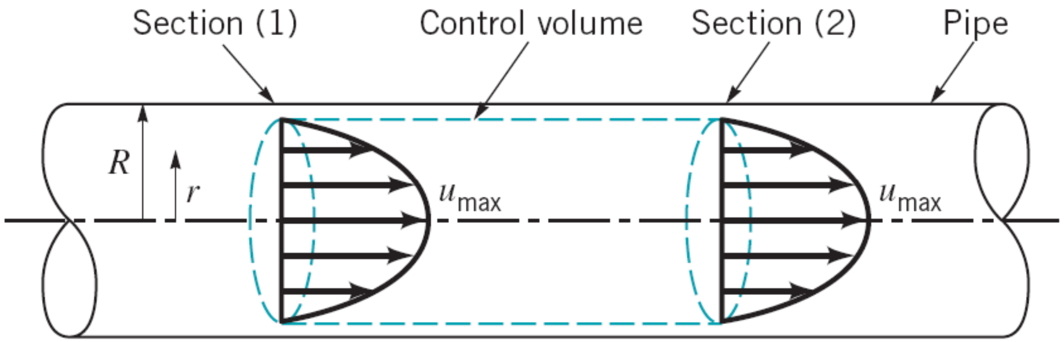
$$\frac{\partial}{\partial t}\int_{CV} e\rho dV + \int_{CS} e\rho \mathbf{V}\cdot\hat{\mathbf{n}} dA = (\dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{net\\in}})_{cv}$$

Power transfer due to normal and tangential stress

- Work transfer rate (i.e. power) can be transferred through a rotating shaft (e.g. turbines, propellers etc)

$$\dot{W}_{shaft} = T_{shaft} w$$ where $\boldsymbol{T_{shaft}}$ – torque and $\boldsymbol{w}$ – angular velocity

- or through the work of normal stress



on a single particle: $$\delta\dot{W}_{\substack{normal\\stress}} = \delta\vec{F}_{\substack{normal\\stress}} \cdot \vec{V} = \sigma\vec{n}\delta A \cdot \vec{V} = -p\vec{V}\cdot\vec{n}\delta A$$

integrating: $$\dot{W}_{\substack{normal\\stress}} = \int_{CS} -p\vec{V}\cdot\vec{n}dA$$

tangential stress: $$\delta\dot{W}_{\substack{tangential\\stress}} = \delta\vec{F}_{\substack{tangential\\stress}} \cdot \vec{V} = 0$$

Power transfer due to normal and tangential stress

- The first law of themrodynamics can be expressed now as:

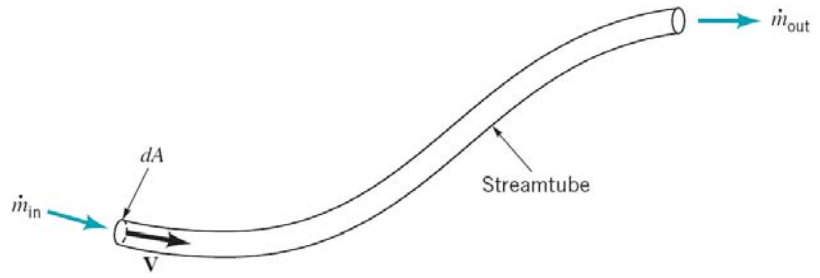
$$\frac{\partial}{\partial t}\int_{CV} e\rho\, d\mathcal{V} + \int_{CS} e\rho \mathbf{V}\cdot\widehat{\mathbf{n}}\, dA = \dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{\text{shaft}\\ \text{net in}}} - \int_{CS} p\vec{V}\cdot\vec{n}\, dA$$

- so, we can obtain the **energy equation**

$$\frac{\partial}{\partial t}\int_{CV} e\rho\, d\mathcal{V} + \int_{CS}\left(\breve{u} + \frac{p}{\rho} + \frac{V^2}{2} + gz\right)\rho \mathbf{V}\cdot\widehat{\mathbf{n}}\, dA = \dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{\text{shaft}\\ \text{net in}}}$$

Application of energy equation

- Let's consider a steady (in the mean, still can be cyclical) flow and take a one stream
- Product **V·n** is non-zero only where liquid crosses the CS; if we have only one stream entering and leaving control volume:



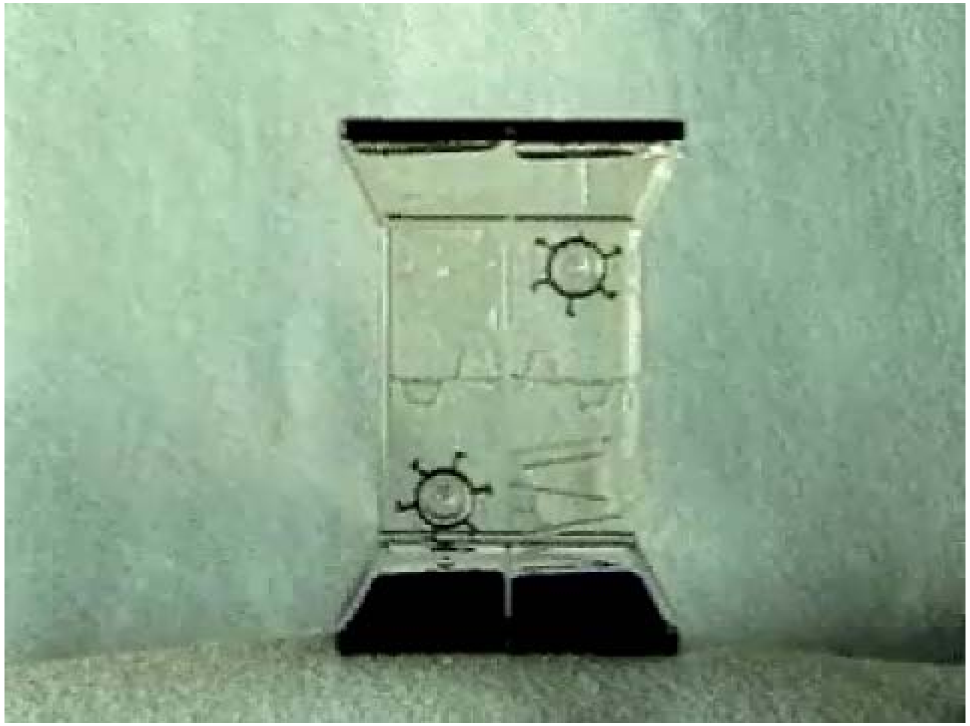
$$\frac{\partial}{\partial t}\int_{CV} e\rho dV + \int_{CS}\left(\breve{u}+\frac{p}{\rho}+\frac{V^2}{2}+gz\right)\rho \mathbf{V}\cdot\hat{\mathbf{n}}dA = \dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{\text{shaft}\\\text{net in}}}$$

$$\int_{CS}\left(\breve{u}+\frac{p}{\rho}+\frac{V^2}{2}+gz\right)\rho \mathbf{V}\cdot\hat{\mathbf{n}}dA = \left(\breve{u}+\frac{p}{\rho}+\frac{V^2}{2}+gz\right)_{out}\dot{m}_{out} - \left(\breve{u}+\frac{p}{\rho}+\frac{V^2}{2}+gz\right)_{in}\dot{m}_{in}$$

$$\dot{m}\left(\breve{u}_{out}-\breve{u}_{in}+\left(\frac{p}{\rho}\right)_{out}-\left(\frac{p}{\rho}\right)_{in}+\left(\frac{V_{out}{}^2-V_{in}{}^2}{2}\right)+g\left(z_{out}-z_{in}\right)\right) = \dot{Q}_{\substack{net\\in}} + \dot{W}_{\substack{\text{shaft}\\\text{net in}}}$$

$$\breve{h} = \breve{u} + \left(\frac{p}{\rho}\right)$$

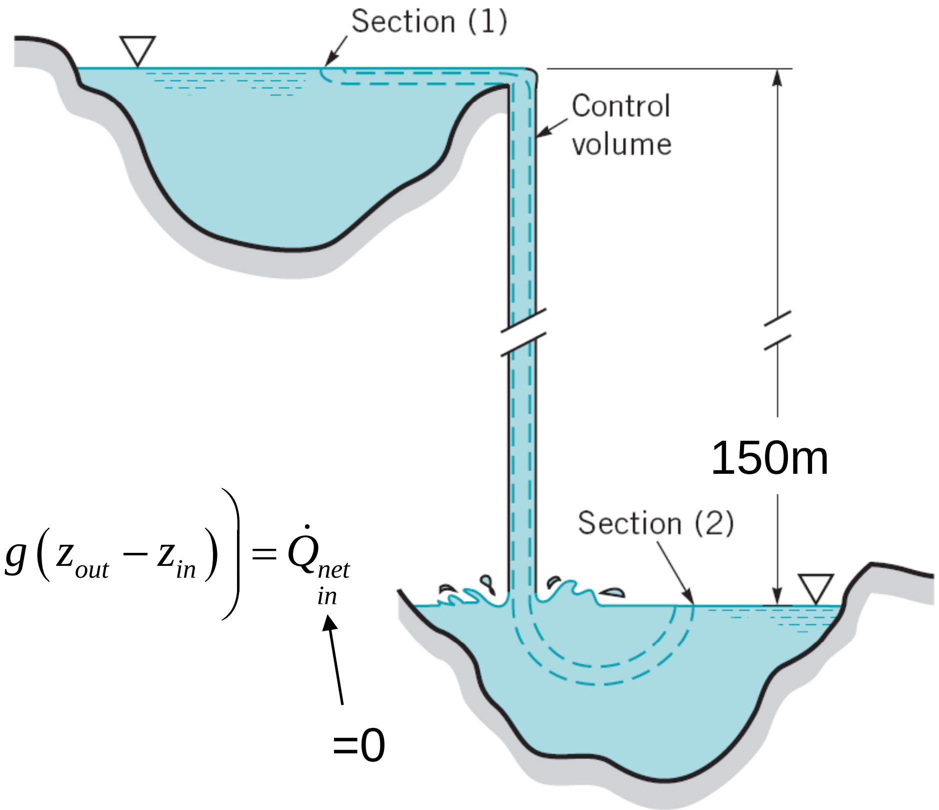
Energy transfer



Work must be done on the device shown to turn it over because the system gains potential energy as the heavy (dark) liquid is raised above the light (clear) liquid. This potential energy is converted into kinetic energy which is either dissipated due to friction as the fluid flows down the ramp or is converted into power by the turbine and then dissipated by friction. The fluid finally becomes stationary again. The initial work done in turning it over eventually results in a very slight increase in the system temperature

Example: Temperature change at a water fall

- find the temperature change after a water fall, c_{water}=4.19 kJ/kg·k



$$\dot{m}\left(\breve{u}_{out} - \breve{u}_{in} + \left(\frac{p}{\rho}\right)_{out} - \left(\frac{p}{\rho}\right)_{in} + \left(\frac{V_{out}^{2} - V_{in}^{2}}{2}\right) + g\left(z_{out} - z_{in}\right)\right) = \dot{Q}_{\substack{net \\ in}}$$

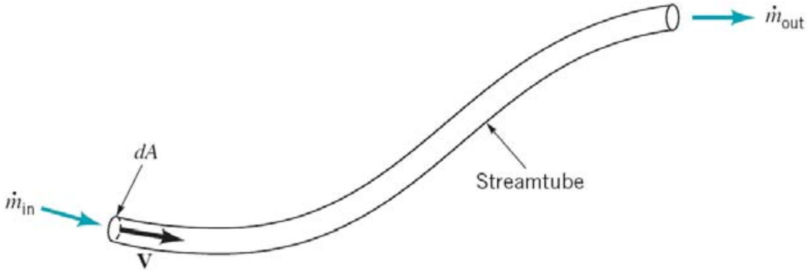
$\left(\frac{p}{\rho}\right)_{out} - \left(\frac{p}{\rho}\right)_{in}$ =0

$\frac{V_{out}^{2} - V_{in}^{2}}{2}$ =0

$\dot{Q}_{\substack{net \\ in}}$ =0

Energy equation vs Bernoulli equation

- Let's return to our one-stream volume, steady flow (also no shaft power)



$$\dot{m}\left(\breve{u}_{out}-\breve{u}_{in}+\left(\frac{p}{\rho}\right)_{out}-\left(\frac{p}{\rho}\right)_{in}+\left(\frac{V_{out}^{2}-V_{in}^{2}}{2}\right)+g\left(z_{out}-z_{in}\right)\right)=\dot{Q}_{\substack{net\\in}}$$

$$\frac{p_{out}}{\rho}+\frac{V_{out}^{2}}{2}+gz_{out}=\frac{p_{in}}{\rho}+\frac{V_{in}^{2}}{2}+gz_{in}-(\breve{u}_{out}-\breve{u}_{in}-q_{\substack{net\\in}}),\qquad q_{\substack{net\\in}}=\frac{\dot{Q}_{\substack{net\\in}}}{\dot{m}}$$

available energy **loss**

- Comparing with Bernoulli equation: $\breve{u}_{out}-\breve{u}_{in}-\dot{q}_{\substack{net\\in}}=0$

i.e. steady incompressible flow should be also ***frictionless***

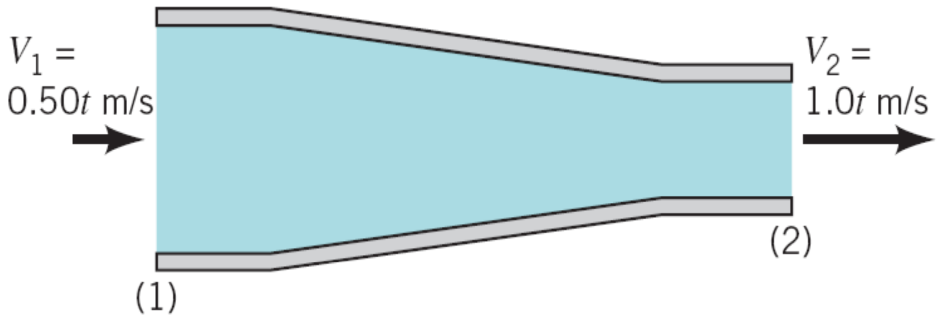
Problems

- 4.4. A velocity field is given by:

$$\vec{V} = x\vec{i} + x(x-1)(y+1)\vec{j}$$

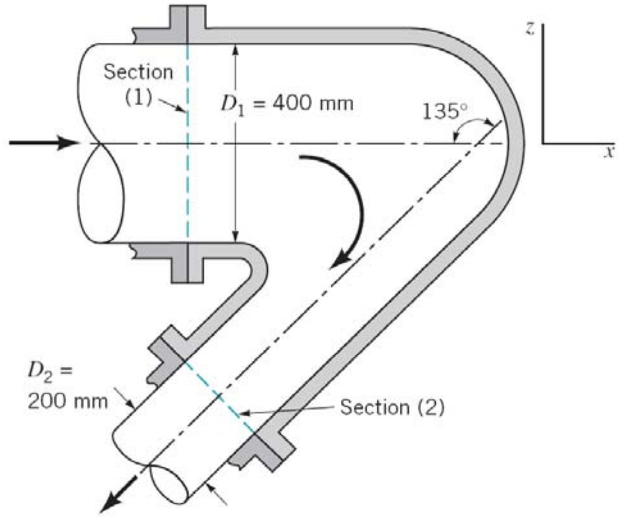
where u and v are in m/s and x,y are in m. Plot the stream lines that passes through x,y=0. Compare this streamline with the streakline through the origin.

- 4.10 Determine local acceleration at points 1 and 2. Is the average convective acceleration between these points negative, zero or positive?



Problems

- **5.19** A converging elbow turns water by 135º. The elbow flow volume is 0.2m³ between sections 1 and 2, water flow rate is 0.4 m³/s, pressures at inlet and outlet 150kPa and 90 kPa, elbow mass 12 kg.



- **5.71** Water flows steadily down the inclined pump. Determine:
  - The pressure difference, p_1-p_2;
  - The loss between sections 1 and 2
  - The axial force exerted on the pipe by water