Pressure in stationary and moving fluid

Lab-On-Chip: Lecture 2

Lecture plan

- what is pressure and how it’s distributed in static fluid
- water pressure in engineering problems
- buoyancy and archimedes law; stability of floating bodies
- fluid kinematics. 2nd Newton law for fluid particles.
- Bernoulli equation and its application

Fluid Statics

- **No shearing stress**
- No relative movement between adjacent fluid particles, i.e. static or moving as a single block
- Main question: How pressure is distributed through the fluid

Pressure at a point

Question: How pressure depends on the orientation of a plane in fluid?

Newton's second law:

$$\sum F_y = p_y \delta x \delta z - p_s \delta x \delta s \sin\theta = \rho \frac{\delta x \delta y \delta z}{2} a_y$$

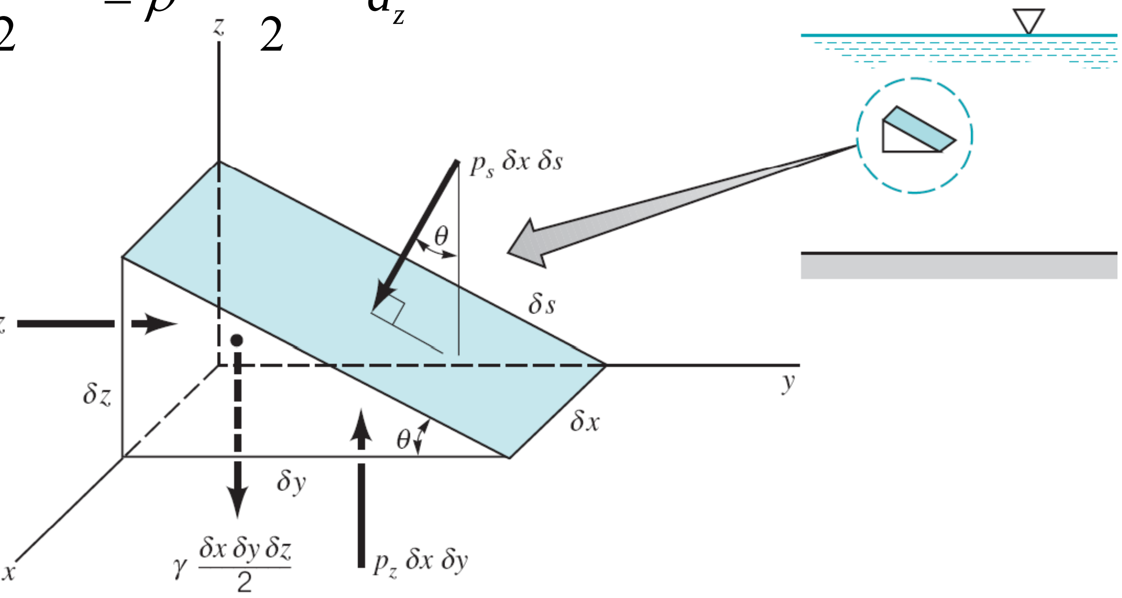
$$\sum F_z = p_z \delta x \delta y - p_s \delta x \delta s \cos\theta - \gamma \frac{\delta x \delta y \delta z}{2} = \rho \frac{\delta x \delta y \delta z}{2} a_z$$

$$\delta y = \delta s \cos\theta \quad \delta z = \delta s \sin\theta$$

$$p_y - p_s = \rho a_y \frac{\delta y}{2}$$

$$p_z - p_s = (\rho a_z + \rho g)\frac{\delta z}{2}$$

$$if \quad \delta x, \delta y, \delta z \to 0, \ p_y = p_s, p_z = p_s$$



- **Pascal's law:** pressure doesn't depend on the orientation of plate (i.e. a scalar number) as long as there are no shearing stresses

Basic equation for pressure field

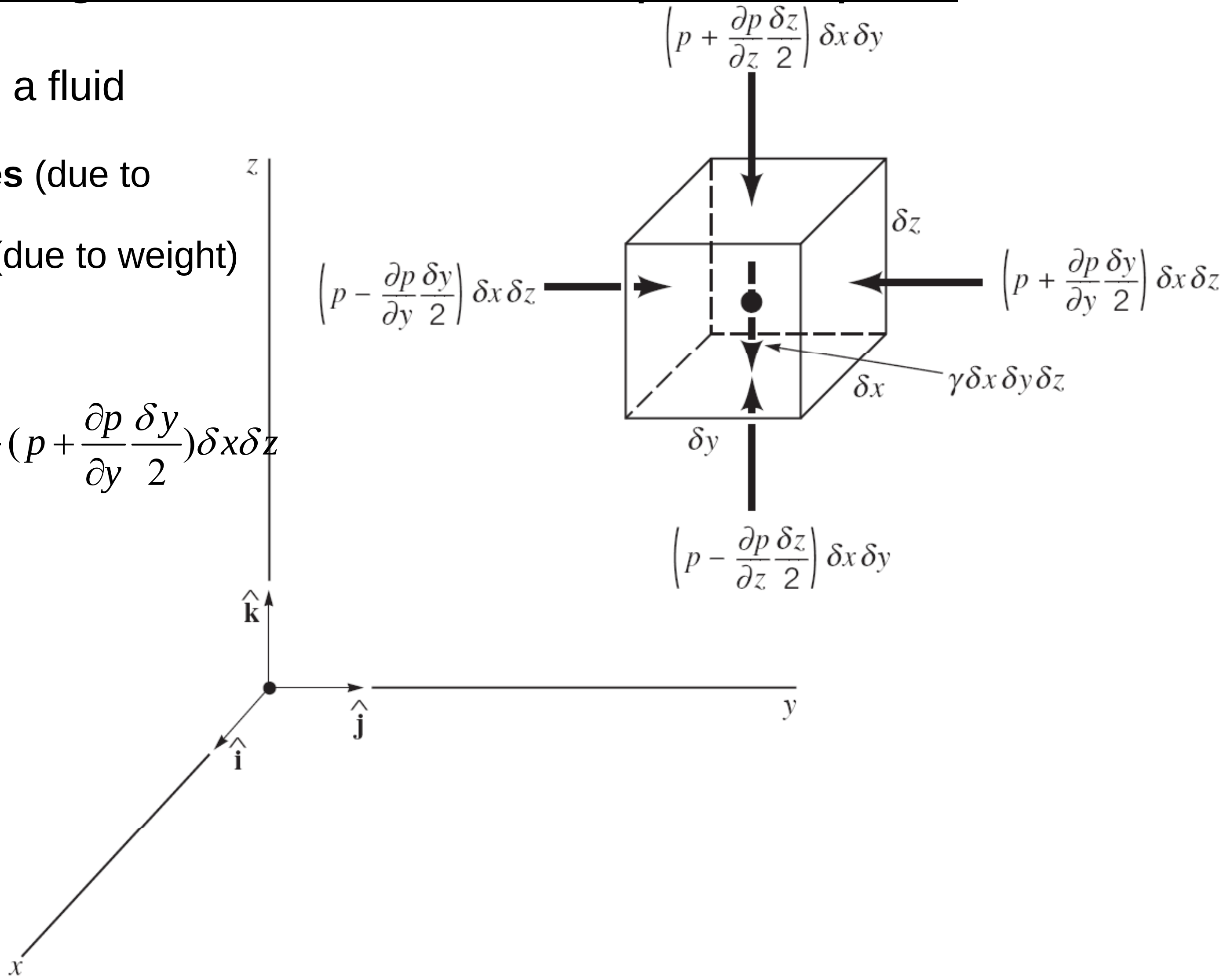
Question: What is the pressure distribution in liquid in absence shearing stress variation from point to point

- Forces acting on a fluid element:
  - **Surface forces** (due to pressure)
  - **Body forces** (due to weight)

Surface forces:

$$\delta F_y = (p - \frac{\partial p}{\partial y}\frac{\delta y}{2})\delta x \delta z - (p + \frac{\partial p}{\partial y}\frac{\delta y}{2})\delta x \delta z$$

$$\begin{cases} \delta F_y = -\dfrac{\partial p}{\partial y}\delta y \delta x \delta z \\ \delta F_x = -\dfrac{\partial p}{\partial x}\delta y \delta x \delta z \\ \delta F_z = -\dfrac{\partial p}{\partial z}\delta y \delta x \delta z \end{cases}$$



Basic equation for pressure field

Resulting surface force in vector form: $$\delta\vec{F} = -\left(\frac{\partial p}{\partial x}\vec{i} + \frac{\partial p}{\partial y}\vec{j} + \frac{\partial p}{\partial z}\vec{k}\right)\delta y\delta x\delta z$$

If we define a gradient as: $$\nabla = \frac{\partial}{\partial x}\vec{i} + \frac{\partial}{\partial y}\vec{j} + \frac{\partial}{\partial z}\vec{k} \qquad \frac{\delta\vec{F}}{\delta y\delta x\delta z} = -\nabla p$$

The weight of element is: $$-\delta W\vec{k} = -\rho g\,\delta y\delta x\delta z\,\vec{k}$$

Newton's second law: $$\delta\vec{F} - \delta W\vec{k} = \delta m\vec{a}$$

$$-\nabla p\delta y\delta x\delta z - \rho g\,\delta y\delta x\delta z\,\vec{k} = -\rho g\,\delta y\delta x\delta z\,\vec{a}$$

General equation of motion for a fluid without shearing stresses

$$\boxed{-\nabla p - \rho g\vec{k} = -\rho g\vec{a}}$$

Pressure variation in a fluid at rest

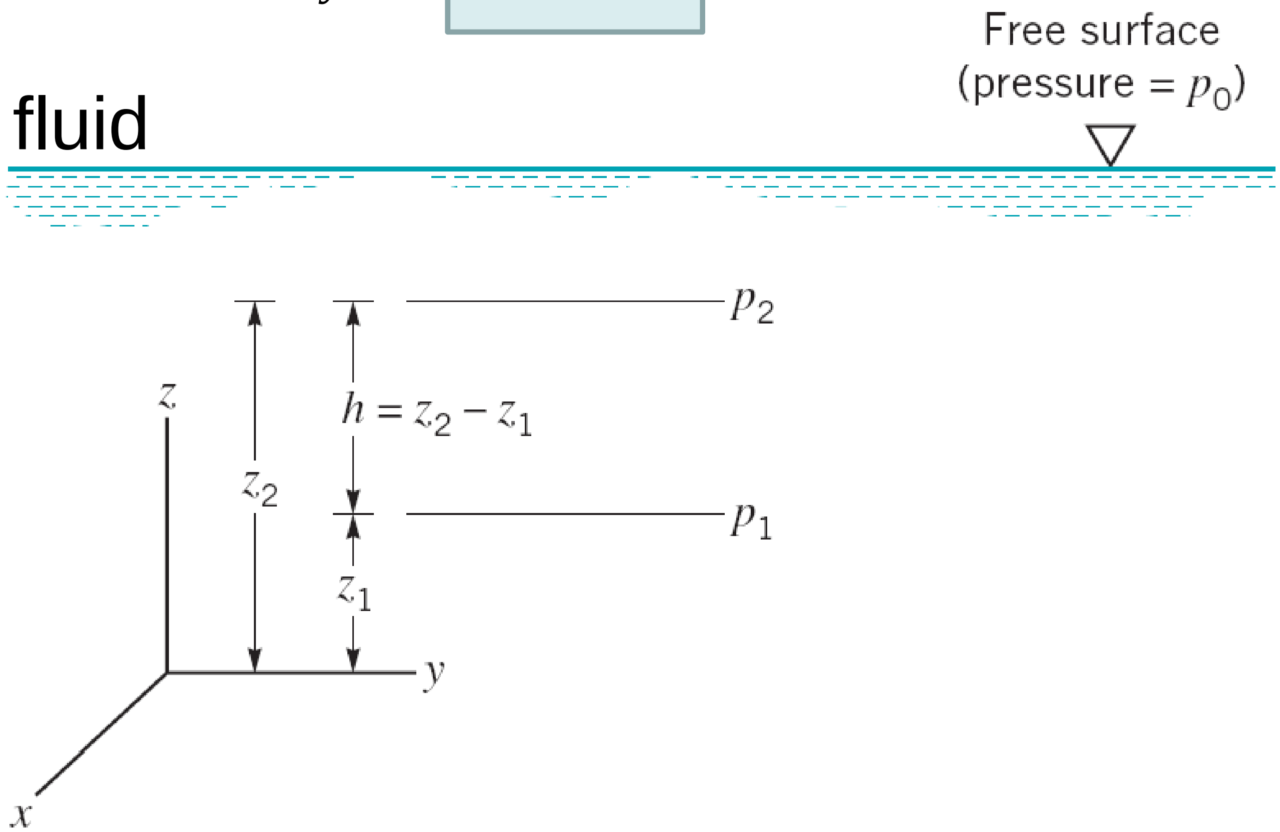
- At rest a=0

$$-\nabla p - \rho g\vec{k} = 0$$

$$\frac{\partial p}{\partial x} = 0 \quad \frac{\partial p}{\partial y} = 0 \quad \boxed{\frac{\partial p}{\partial z} = -\rho g}$$

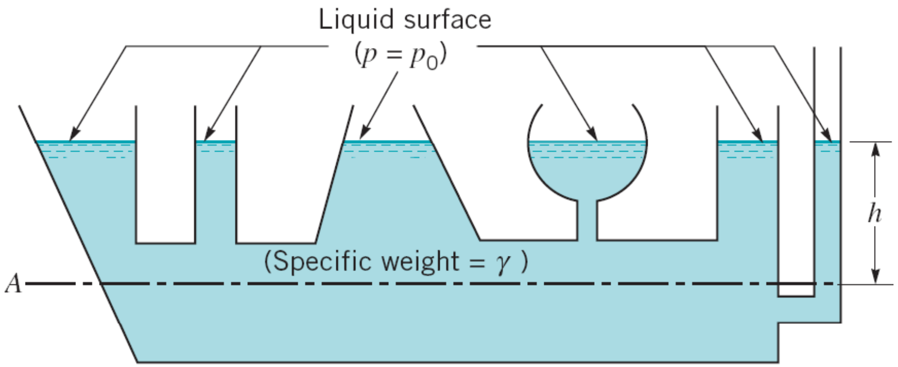
- Incompressible fluid

$$p_1 = p_2 + \rho g h$$

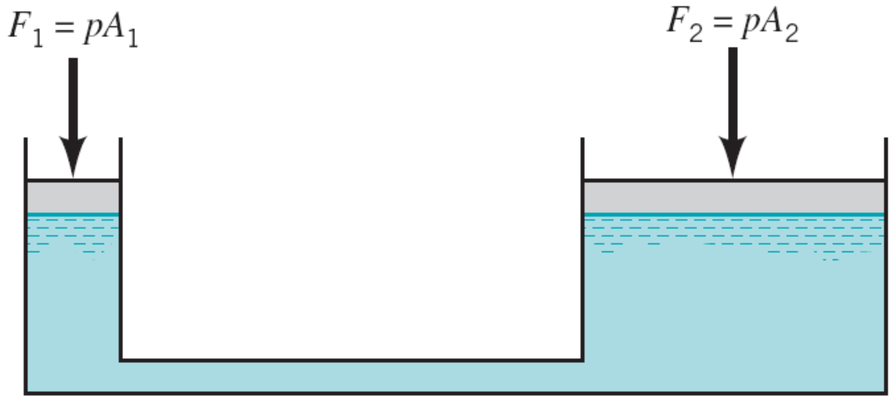


Fluid statics

Same pressure –
much higher force!



Fluid equilibrium



Transmission of fluid pressure, e.g. in hydraulic lifts

- Pressure depends on the depth in the solution not on the lateral coordinate

Compressible fluid

- Example: let's check pressure variation in the air (in atmosphere) due to compressibility:
  - Much lighter than water, 1.225 kg/m^3 against 1000kg/m^3 for water
  - Pressure variation for small height variation are negligible
  - For large height variation compressibility should be taken into account:

$$p = \frac{nRT}{V} = \rho RT$$

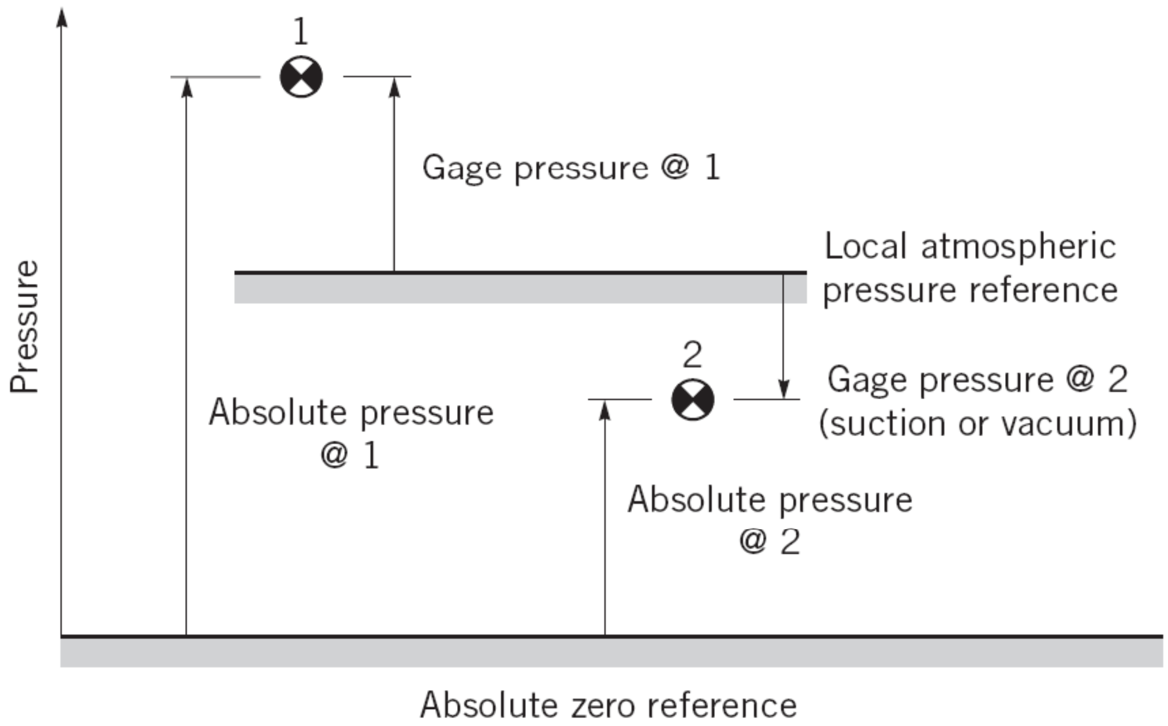
$$\frac{dp}{dz} = -\rho g = -\frac{gp}{RT}$$

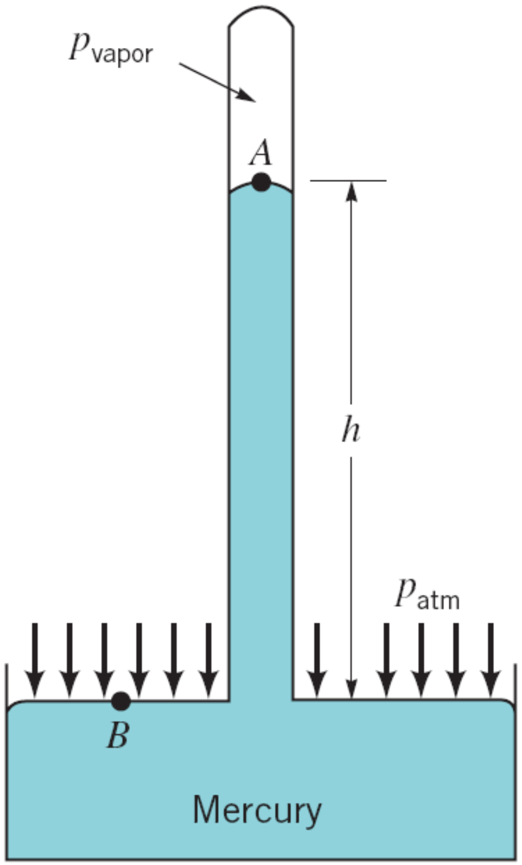
~8 km

$$\text{assuming } T = const \Rightarrow p_2 = p_1 \exp\left[\frac{g(z_1 - z_2)}{RT_0}\right];\ p(h) = p_0 e^{-h/H}$$

Measurement of pressure

- Pressures can be designated as **absolute** or **gage (gauge) pressures**



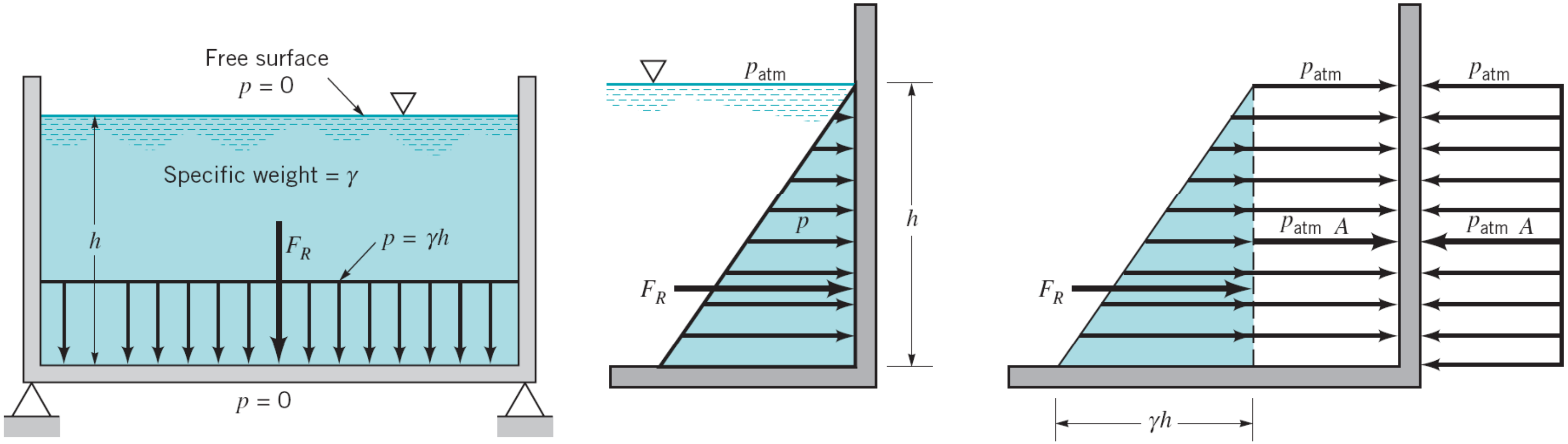


$$p_{atm} = \gamma h + p_{vapor}$$

very small!

Hydrostatic force on a plane surface

- For fluid in rest, there are no shearing stresses present and the force must be **perpendicular** to the surface.
- Air pressure acts on both sides of the wall and will cancel.



Force acting on a side wall in rectangular container: $F_R = p_{av} A = \rho g \frac{h}{2} bh$

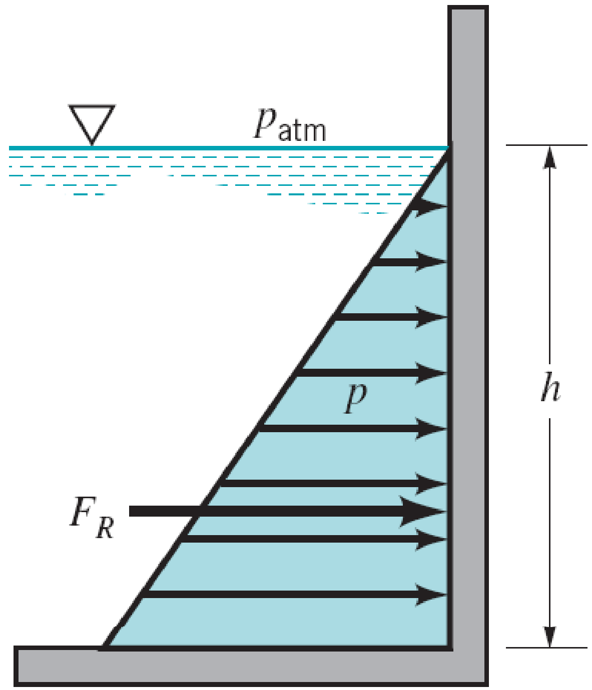
Example: Pressure force and moment acting on aquarium walls

- Force acting on the wall

$$F_R = \int_A \rho g h \, dA = \int_0^H \rho g (H-y) \cdot b dy = \rho g \frac{H^2}{2} b$$

- Generally: $F_R = \rho g \sin\theta \int_A y dA = \rho g \sin\theta \, y_c A$

Centroid (first moment of the area)



- Momentum of force acting on the wall

$$F_R y_R = \int_A \rho g h \, y dA = \int_0^H \rho g (H-y) y \cdot b dy = \rho g \frac{H^3}{6} b$$

$$y_R = H/3$$

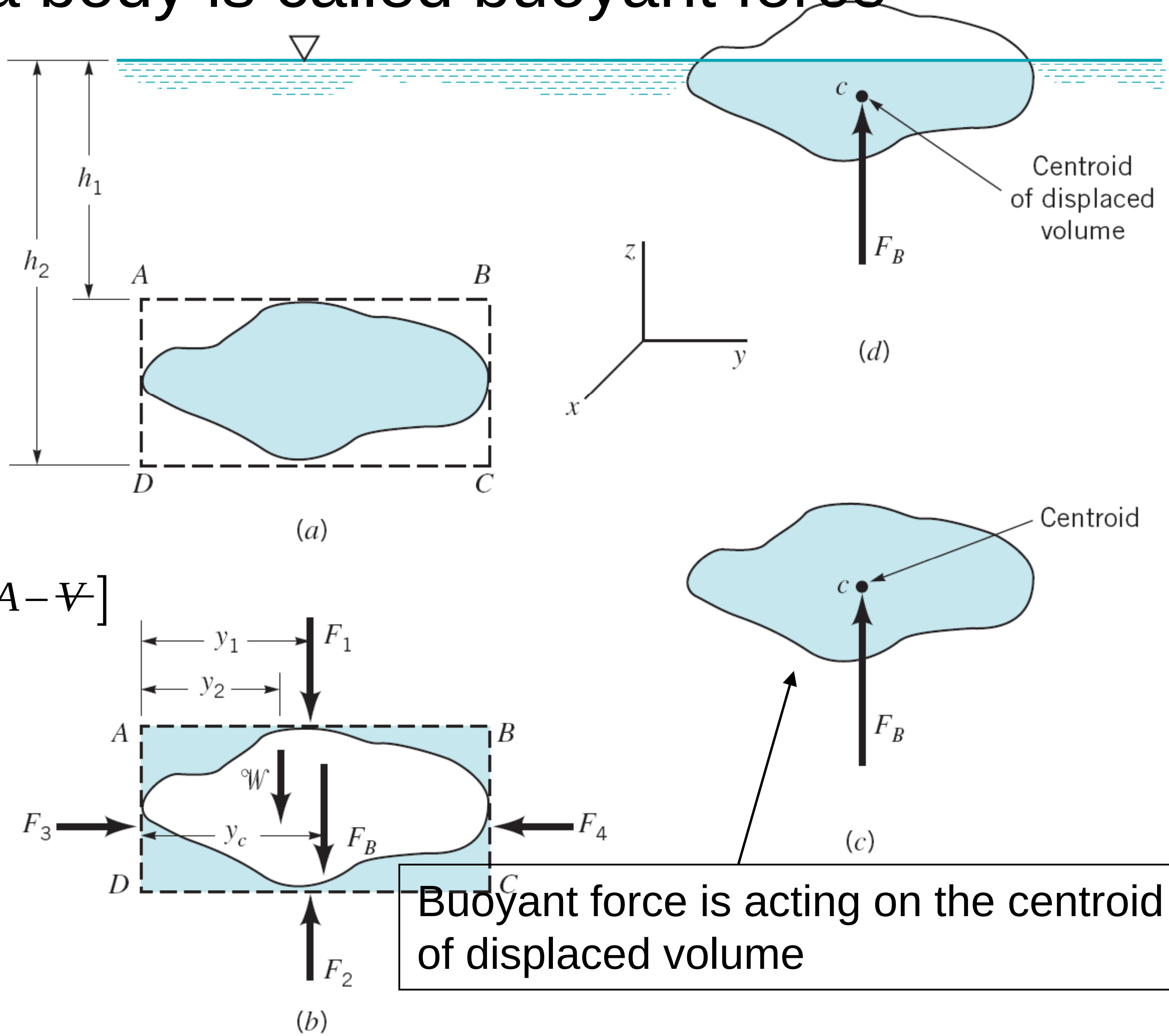
- Generally,

$$y_R = \frac{\int_A y^2 dA}{y_c A}$$

Pressure force on a curved surface

Buoyant force: Archimedes principle

- when a body is totally or partially submerged a fluid force acting on a body is called buoyant force



$$F_B = F_2 - F_1 - W$$

$$F_2 - F_1 = \rho g (h_2 - h_1) A$$

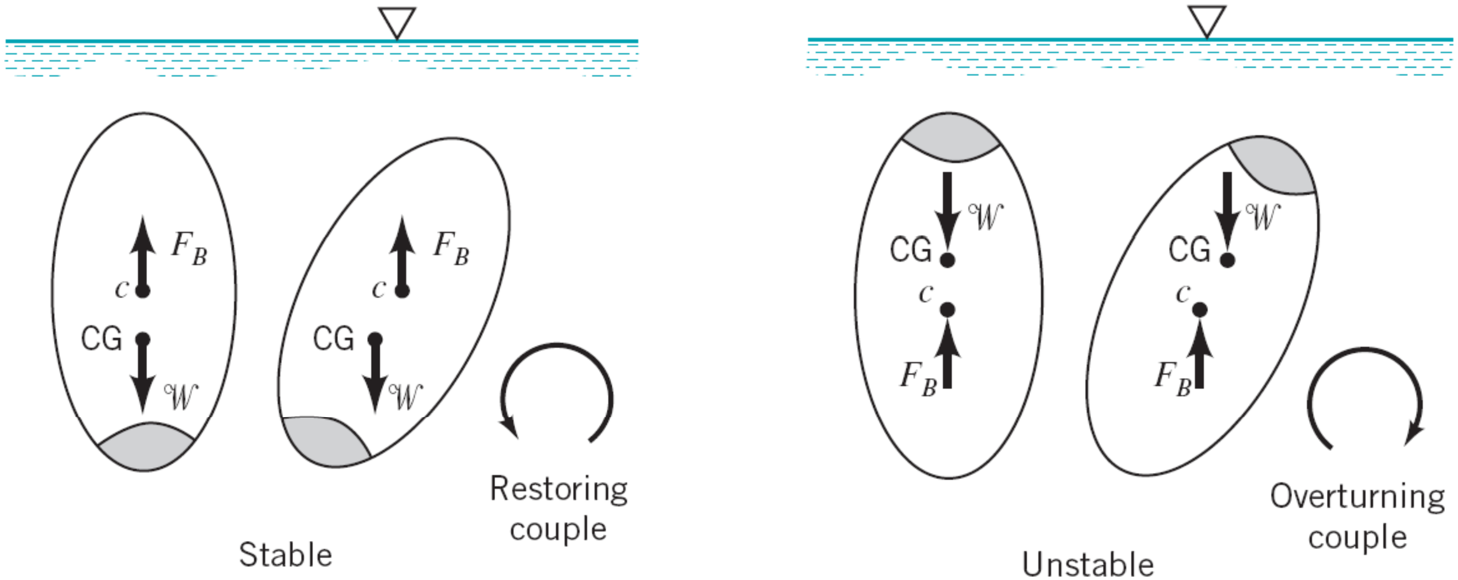
$$F_B = \rho g (h_2 - h_1) A - \rho g \left[(h_2 - h_1) A - \mathcal{V} \right]$$

$$\boxed{F_B = \rho g \mathcal{V}}$$

Buoyant force is acting on the centroid of displaced volume

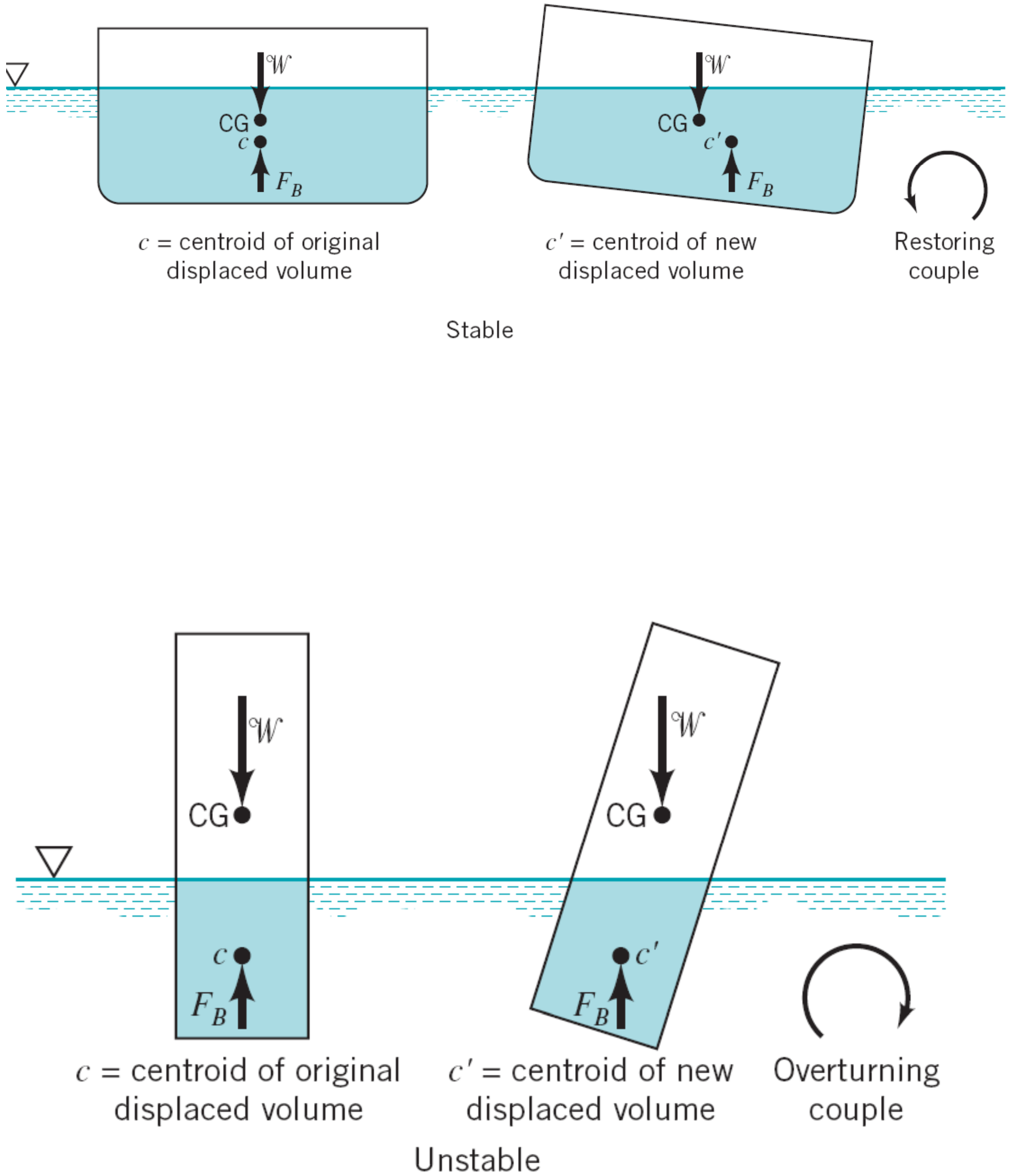
Stability of immersed bodies

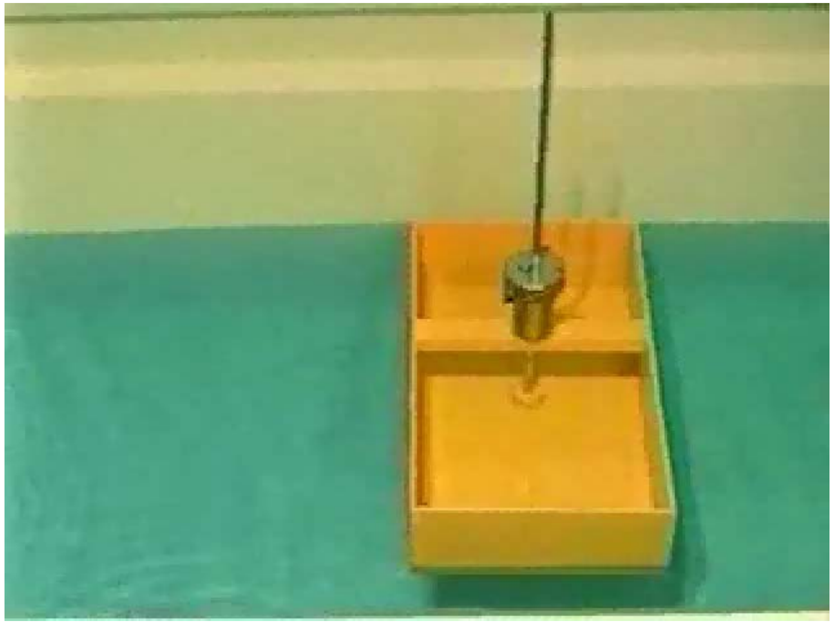
- Totally immersed body



Stability of immersed bodies

- Floating body



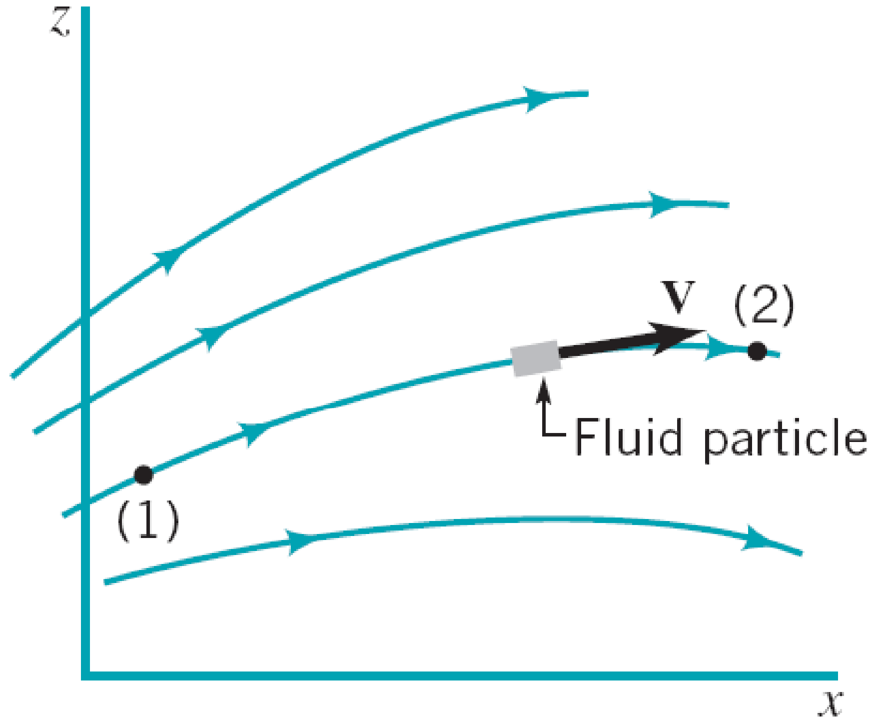


Elementary fluid dynamics: Bernoulli equation

Bernuolli equation – ”the most used and most abused equation in fluid mechanics”

<u>Assumptions:</u>

- steady flow: each fluid particle that passes through a given point will follow the same path
- inviscid liquid (no viscosity, therefore no thermal conductivity

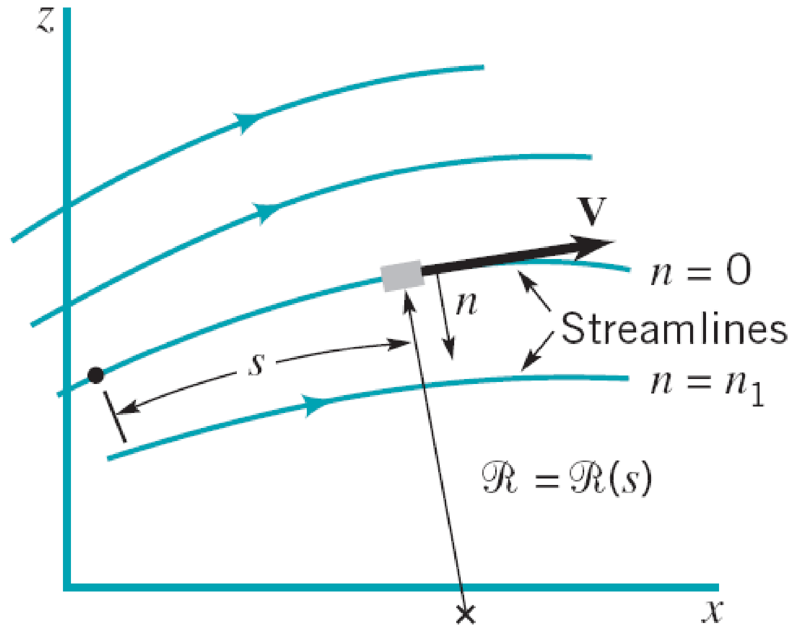


$\boxed{\mathbf{F} = m\mathbf{a}}$

Net pressure force + Net gravity force

Streamlines

<u>Streamlines</u>: the lines that are tangent to velocity vector through the flow field

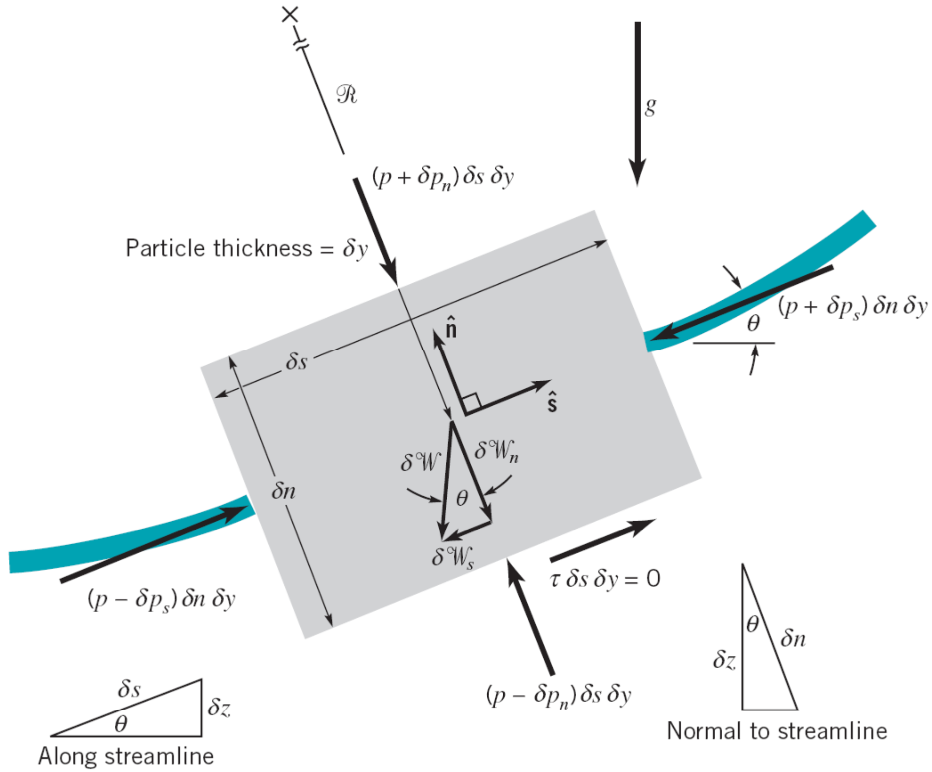


Acceleration along the streamline:

$$a_s = \frac{dv}{dt} = \frac{\partial v}{\partial s}\frac{\partial s}{\partial t} = \frac{\partial v}{\partial s} v$$

Centrifugal acceleration:

$$a_n = \frac{v^2}{R}$$



Along the streamline

$$\sum \delta F_s = m a_s = \rho \delta V \frac{\partial v}{\partial s} v$$

$$\sum \delta F_s = \delta W_s + \delta F_{ps} = -\rho g \delta V \sin(\theta) - \frac{\partial p}{\partial s} \delta V$$

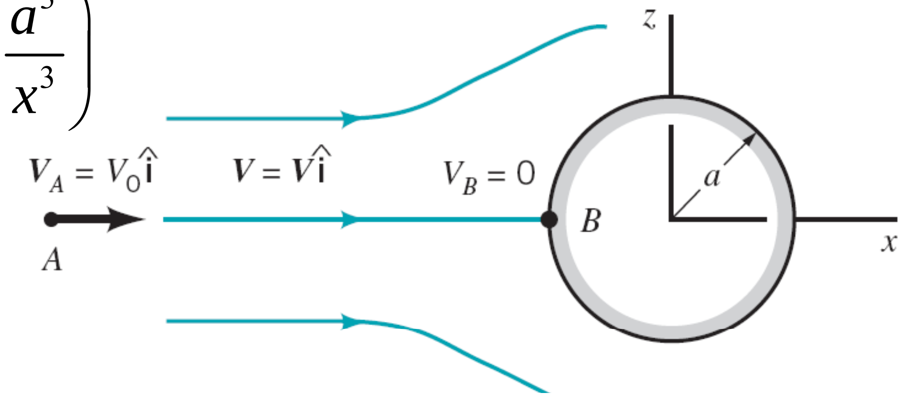
$$-\rho g \sin(\theta) - \frac{\partial p}{\partial s} = \rho \frac{\partial v}{\partial s} v$$

Balancing ball

Pressure variation along the streamline

- Consider inviscid, incompressible, steady flow along the horizontal streamline A-B in front of a sphere of radius a. Determine pressure variation along the streamline from point A to point B. Assume:

$$V = V_0\left(1 + \frac{a^3}{x^3}\right)$$



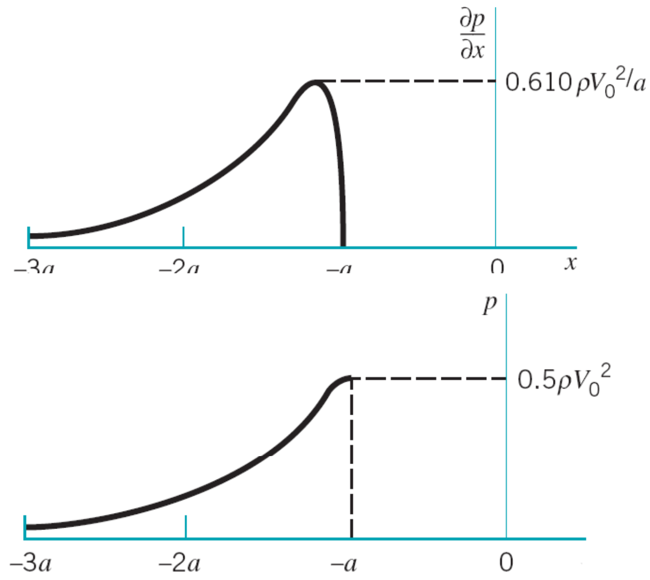
Equation of motion:

$$\frac{\partial p}{\partial s} = -\rho v \frac{\partial v}{\partial s}$$

$$v\frac{\partial v}{\partial s} = -3v_0^2\left(1 + \frac{a^3}{x^3}\right)\frac{a^3}{x^4}$$

$$\frac{\partial p}{\partial x} = \frac{3\rho a^3 v_0^2}{x^4}\left(1 + \frac{a^3}{x^3}\right)$$

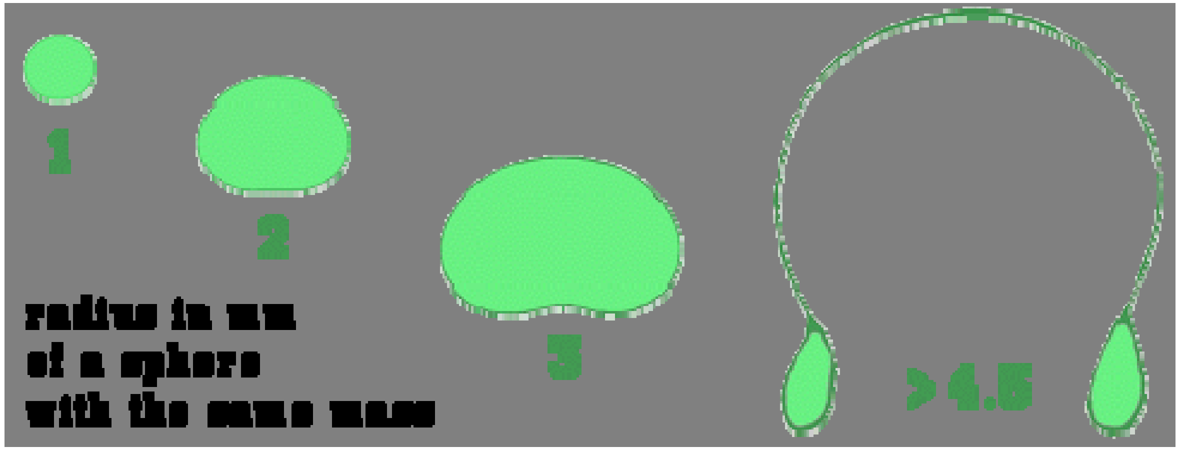
$$\Delta p = \int_{-\infty}^{-a} \frac{3\rho a^3 v_0^2}{x^4}\left(1 + \frac{a^3}{x^3}\right)dx = -\rho v_0^2\left(\frac{a^3}{x^3} + \frac{1}{2}\left(\frac{a}{x}\right)^6\right)$$



(c)

Raindrop shape

The actual shape of a raindrop is a result of balance between the surface tension and the air pressure



Bernoulli equation

Integrating $$-\rho g\sin(\theta)-\frac{\partial p}{\partial s}=\rho\frac{\partial v}{\partial s}v$$

$$\frac{dz}{ds}$$

$$\frac{dp}{ds},\ \text{n=const along streamline}$$

$$-\rho g\frac{dz}{ds}-\frac{dp}{ds}=\frac{1}{2}\rho\frac{dv^2}{ds}$$

We find $$dp+\frac{1}{2}\rho d(v^2)+\rho g dz=0$$ Along a streamline

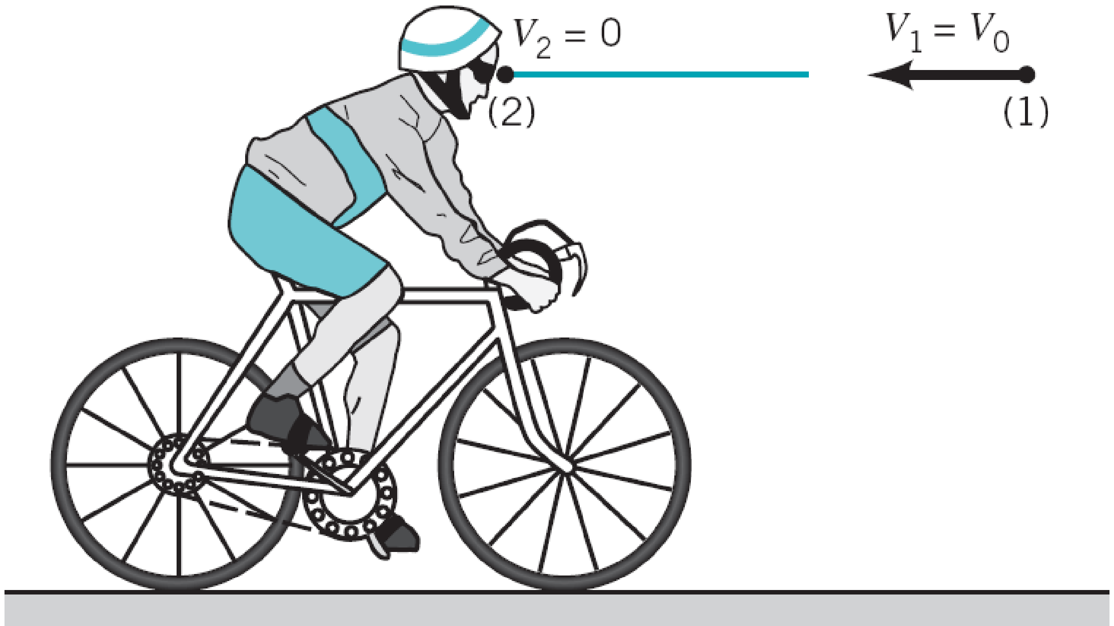
Assuming incompressible flow:

$$p+\frac{1}{2}\rho v^2+\rho g z=const$$ Along a streamline

Bernoulli equation

Example: Bicycle

- Let's consider coordinate system fixed to the bike.
  Now Bernoulli equation can be applied to



$$p_2 - p_1 = \frac{1}{2}\rho v_0^{\ 2}$$

Pressure variation normal to streamline

$$\sum \delta F_n = m a_n = \rho \delta V \frac{v^2}{R}$$

$$\sum \delta F_n = \delta W_n + \delta F_{pn} = -\rho g \delta V \cos(\theta) - \frac{\partial p}{\partial n} \delta V$$

$$-\rho g \frac{dz}{dn} - \frac{\partial p}{\partial n} = \rho \frac{v^2}{R}$$

$$p + \rho \int \frac{v^2}{R} dn + \rho g z = const$$

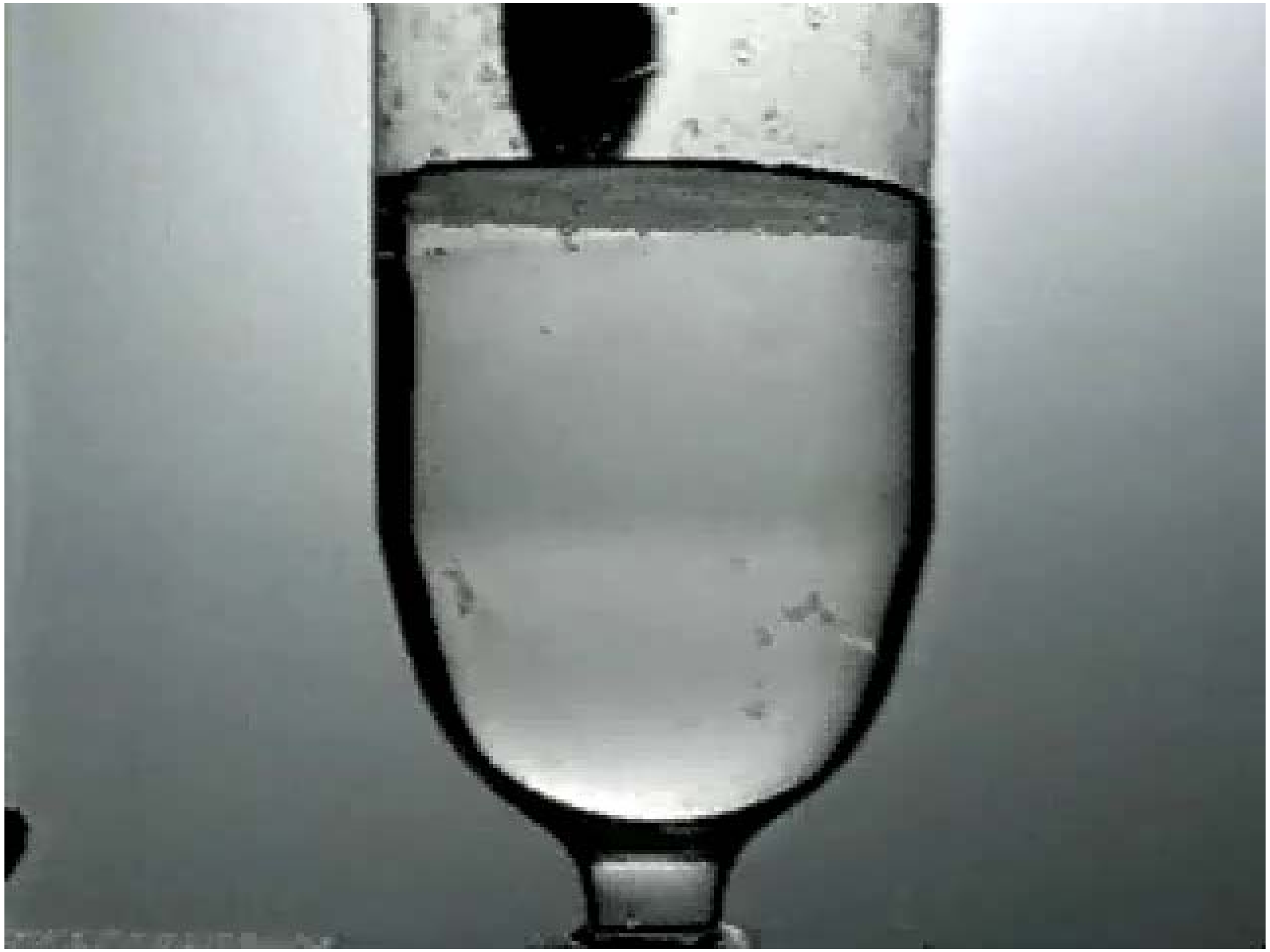
<u>Across streamlines</u>

compare

$$p + \frac{1}{2} \rho v^2 + \rho g z = const$$

<u>Along a streamline</u>

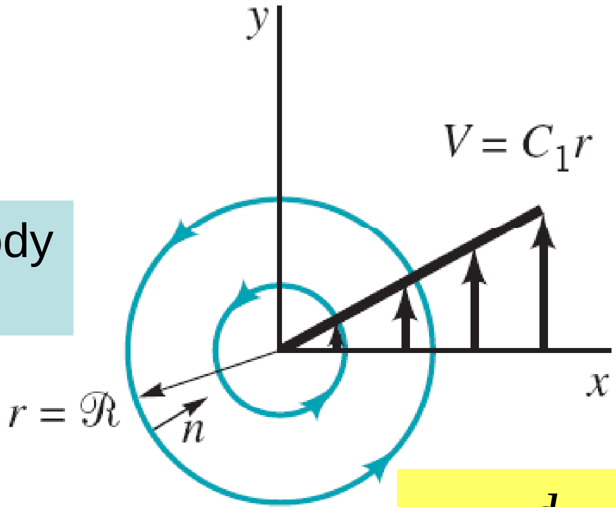
Free vortex



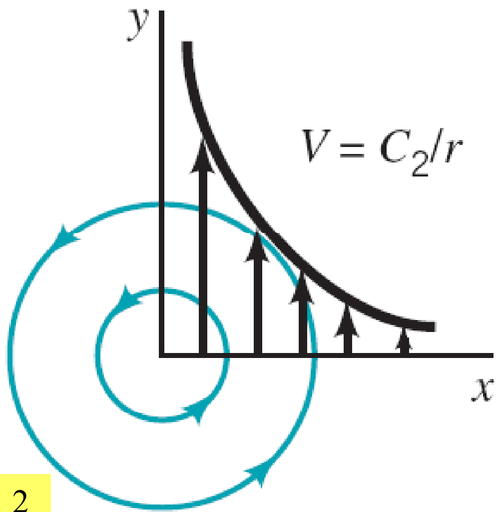
Example: pressure variation normal to streamline

- Let's consider 2 types of vortices with the velocity distribution as below:

solid body rotation



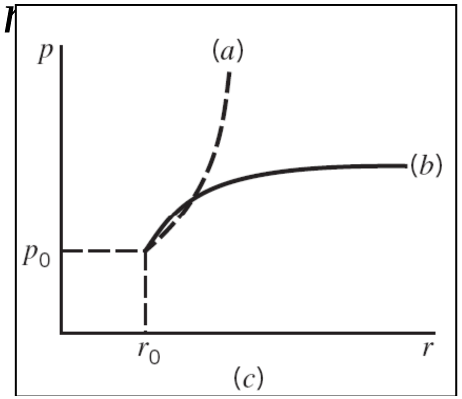
$$-\rho g\frac{dz}{dn}-\frac{\partial p}{\partial n}=\rho\frac{v^2}{R}$$



free vortex

as $\dfrac{\partial}{\partial n}=-\dfrac{\partial}{\partial r}$, $\dfrac{\partial p}{\partial r}=\dfrac{\rho V^2}{r}=\rho C_1^{\,2} r$

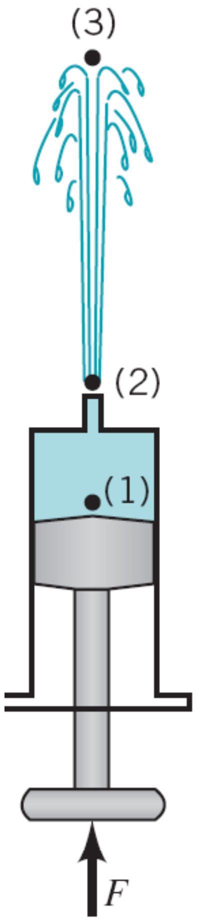
$$p=\frac{1}{2}\rho C_1^{\,2}\left(r_0^{\,2}-r^2\right)+p_0$$



$$\frac{\partial p}{\partial r}=\frac{\rho V^2}{r}=\rho\frac{C_1^{\,2}}{r^3}$$

$$p=\frac{1}{2}\rho C_2^{\,2}\left(\frac{1}{r_0^{\,2}}-\frac{1}{r^2}\right)+p_0$$

$$p + \frac{1}{2}\rho v^2 + \rho g z = const$$



Point	Energy Type		
	Kinetic $\rho V^2/2$	Potential γz	Pressure p
1	Small	Zero	Large
2	Large	Small	Zero
3	Zero	Large	Zero

Static, Stagnation, Dynamic and TotalPressure

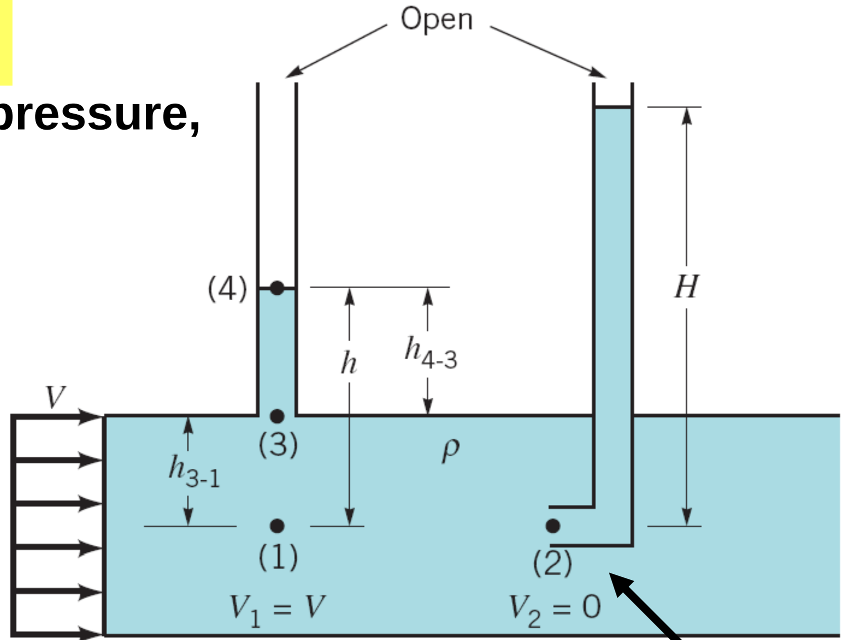
- each term in Bernoulli equation has dimensions of pressure and can be interpreted as some sort of pressure

$$p + \frac{1}{2}\rho v^2 + \rho g z = const$$

hydrostatic pressure,

dynamic pressure,

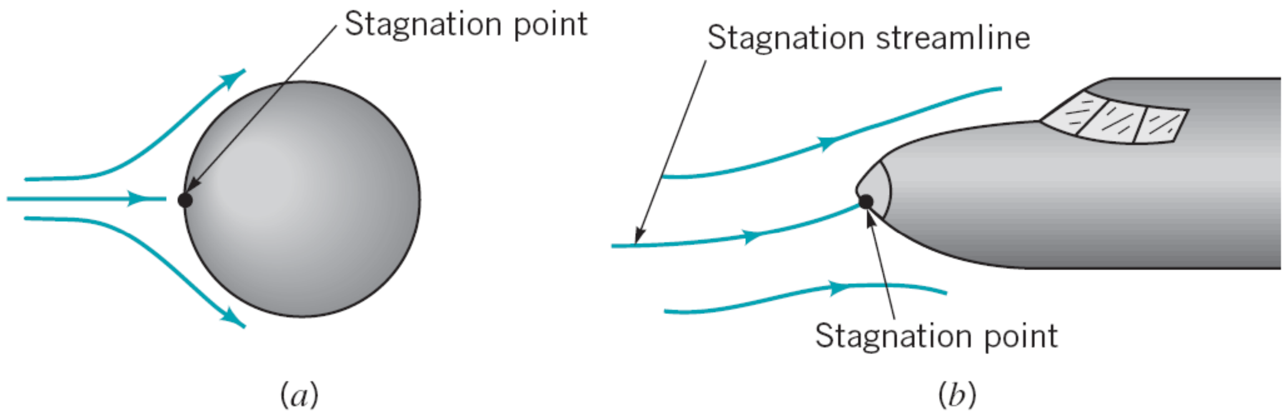
static pressure,
point (3)

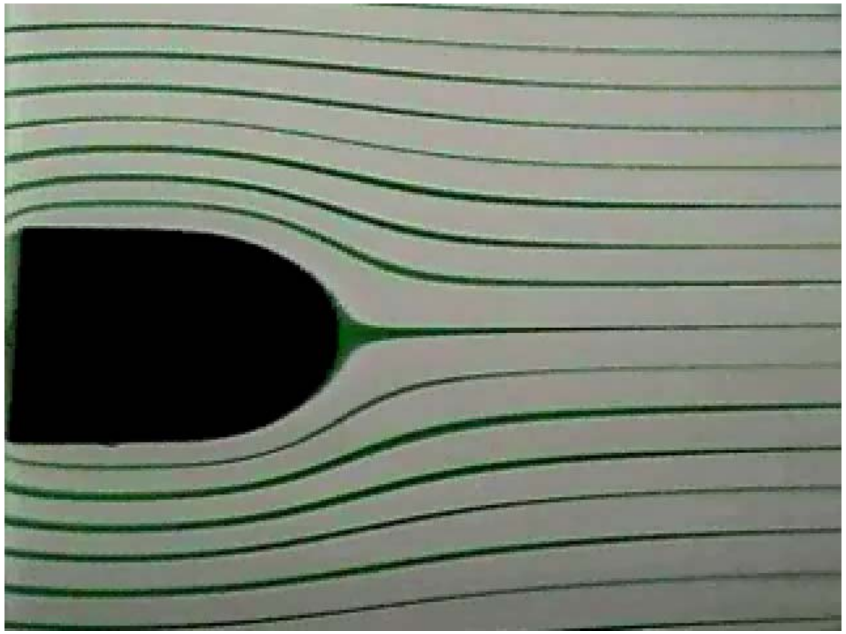


Velocity can be determined from stagnation pressure:

$$p_2 = p_1 + \frac{1}{2}\rho v^2$$

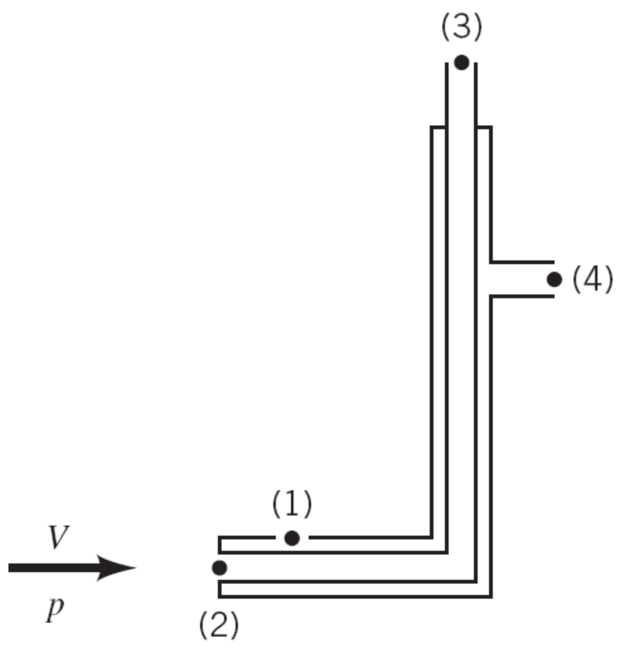
Stagnation pressure

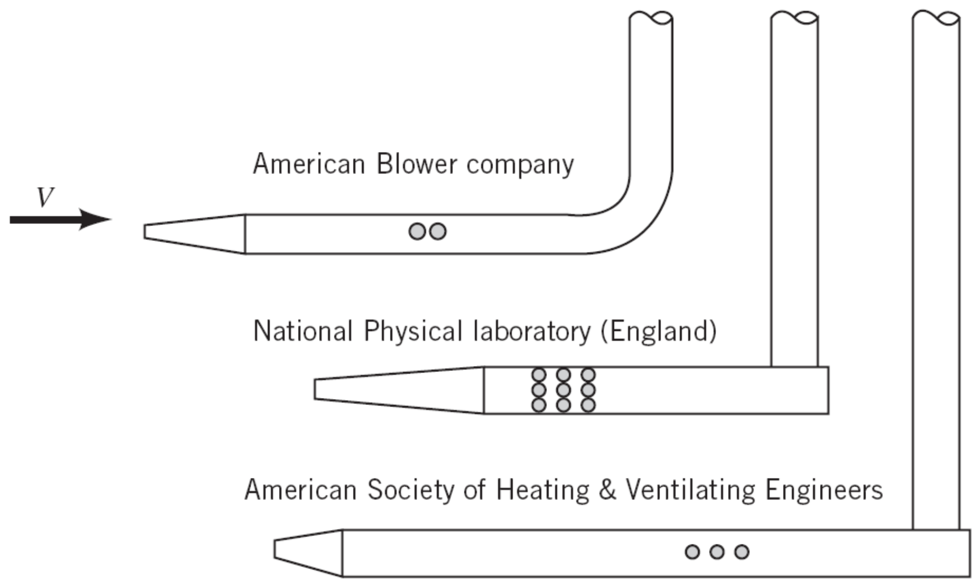




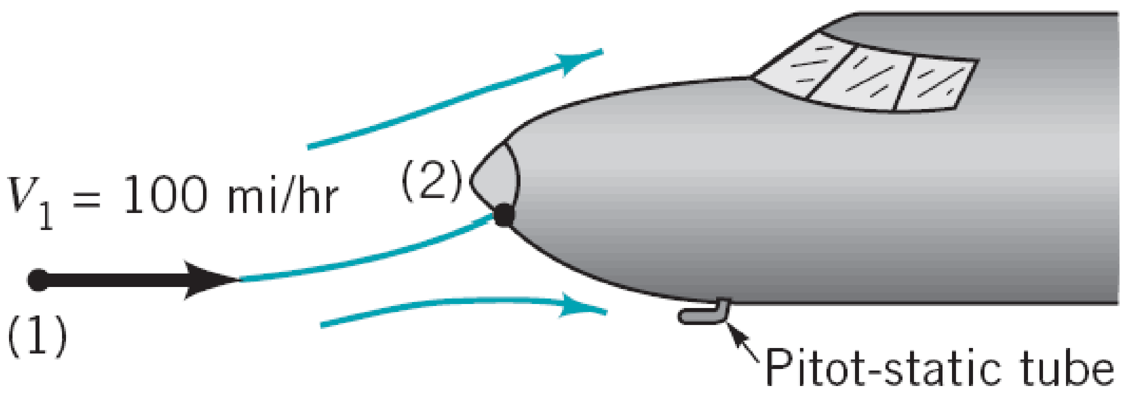
On any body in a flowing fluid there is a stagnation point. Some of the fluid flows "over" and some "under" the body. The dividing line (the stagnation streamline) terminates at the stagnation point on the body.
As indicated by the dye filaments in the water flowing past a streamlined object, the velocity decreases as the fluid approaches the stagnation point. The pressure at the stagnation point (the stagnation pressure) is that pressure obtained when a flowing fluid is decelerated to zero speed by a frictionless process

Pitot-static tube

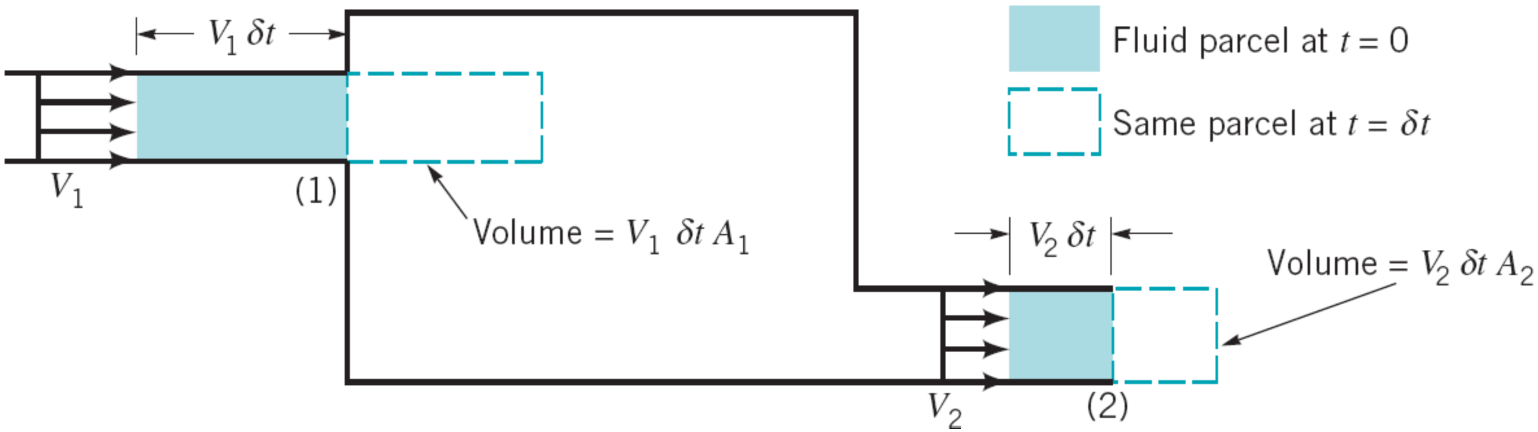




■ FIGURE 3.7 **Typical Pitot-static tube designs.**

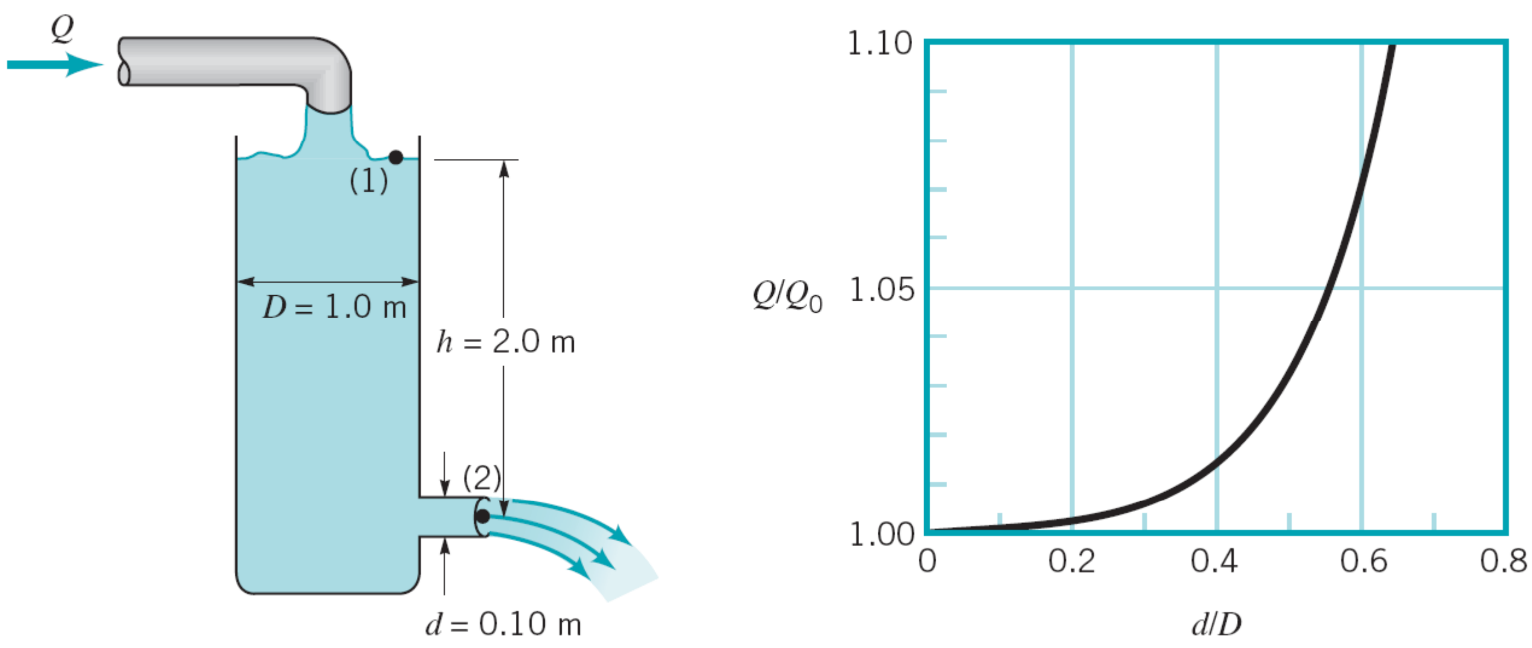


Steady flow into and out of a tank.



$$\boxed{\rho_1 A_1 v_1 = \rho_2 A_2 v_2}$$

Determine the flow rate to keep the height constant

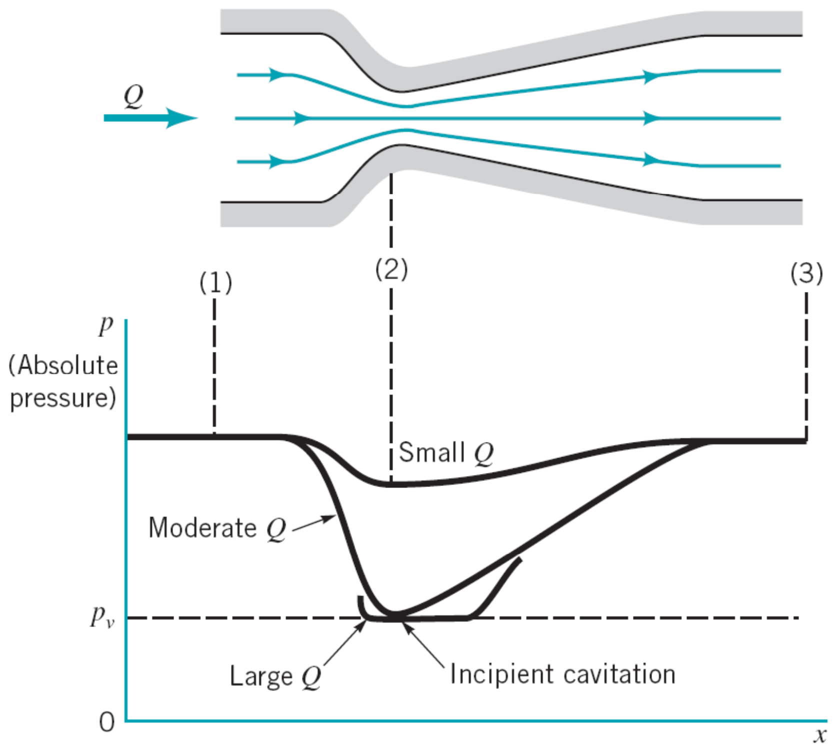


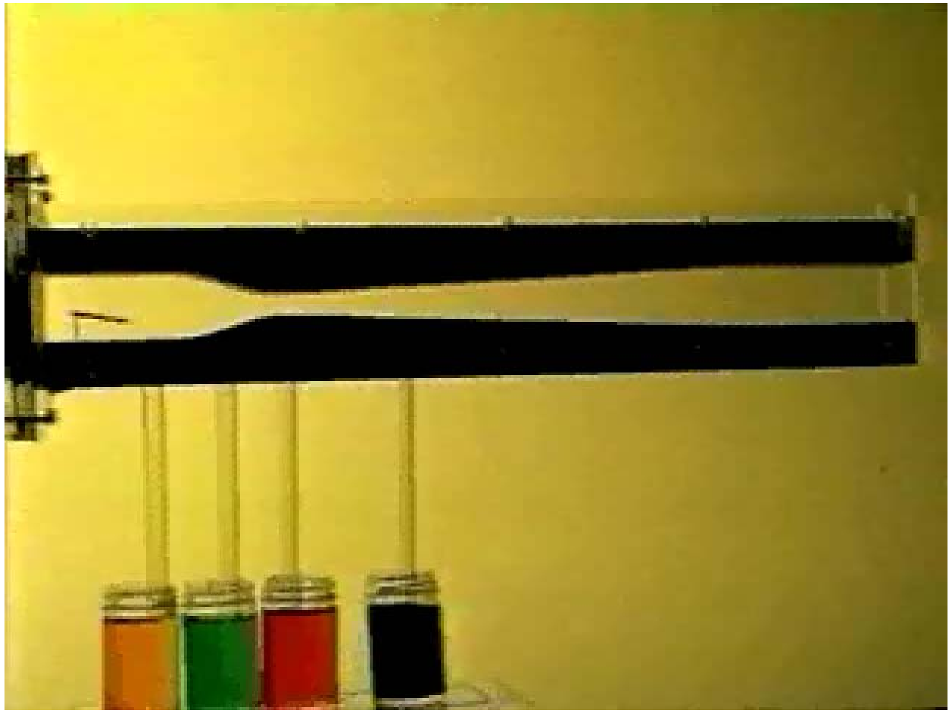
(a) (b)

$$p_1 + \frac{1}{2}\rho v_1^{\,2} + \rho g z_1 = p_2 + \frac{1}{2}\rho v_2^{\,2} + \rho g z_2$$

$$Q = A_1 v_1 = A_2 v_2$$

Venturi channel

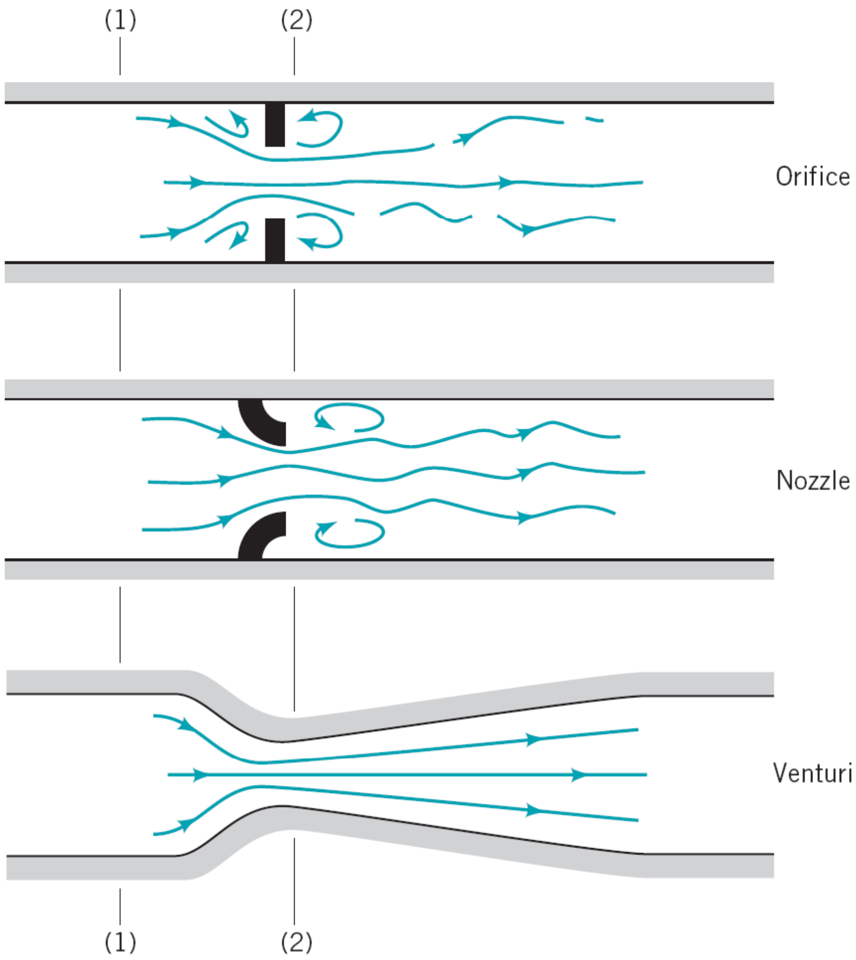




Measuring flow rate in pipes

$$p_1 + \frac{1}{2}\rho v_1{}^2 = p_2 + \frac{1}{2}\rho v_2{}^2$$

$$Q = A_1 v_1 = A_2 v_2$$

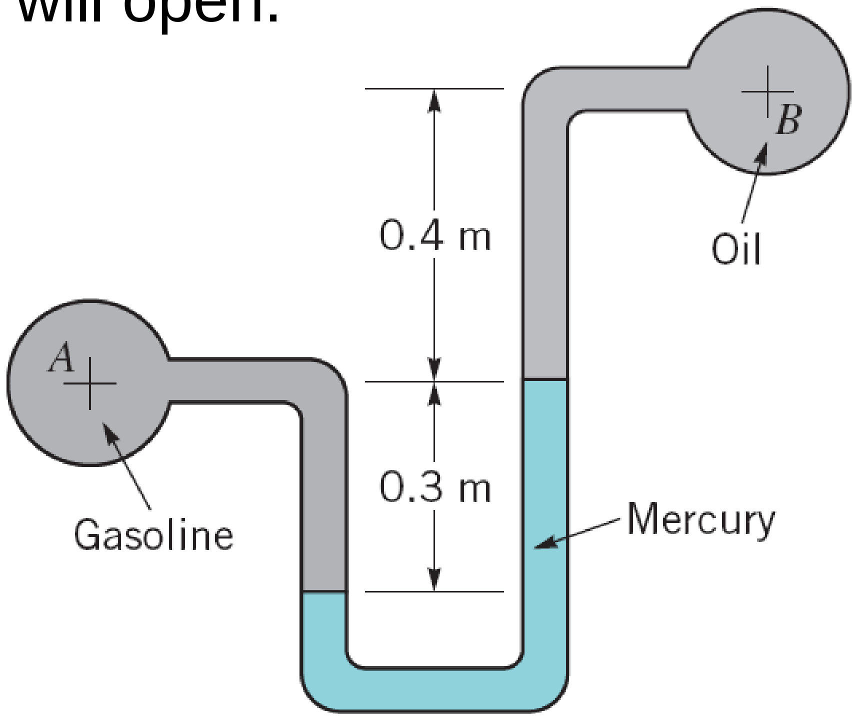


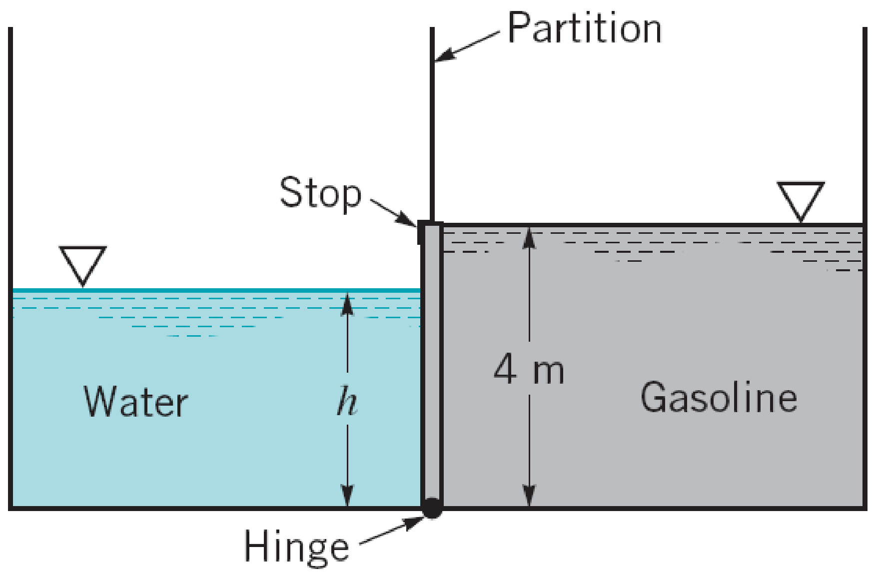
Restriction on use of Bernoulli equation

- Incompressible flow
- Steady flow
- Application of Bernoulli equation across the stream line is possible only in irrotational flow
- Energy should be conserved along the streamline (inviscid flow + no active devices).

Probelms

- **2.24** Pipe A contains gasoline (SG=0.7), pipe B contains oil (SG=0.9). Determine new differential reading of pressure in A decreased by 25 kPa. The initial differential reading is 30cm as shown.
- **2.39** An open tank contains gasoline ρ=700kg/cm at a depth of 4m. The gate is 4m high and 2m wide. Water is slowly added to the empty side of the tank. At what depth *h* the gate will open.





Problems

- **3.29** The circular stream of water from a faucet is observed to taper from a diameter 20 mm (at the faucet) down to 10 mm in a distance of 50 cm. Determine the flow rate.

- Water flows through a pipe contraction as shown below. Calculate flowrate as a function of smaller pipe diameter for both manometer configuration