Lecture 4

Equation of motions for a liquid.
Inviscid and Viscous flow

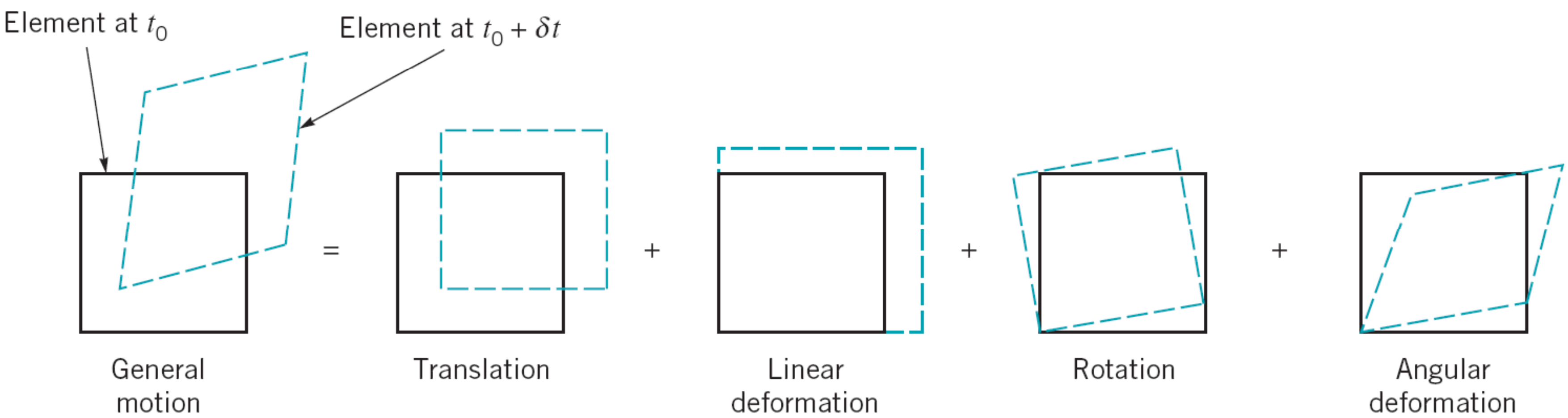
Navier-Stockes equation

Differential analysis of Fluid Flow

- The aim: to produce differential equation describing the motion of fluid in detail

Fluid Element Kinematics

- Any fluid element motion can be represented as consisting of translation, linear deformation, rotation and angular deformation



Velocity and acceleration field

- Velocity field

$$\boldsymbol{V} = u\hat{\boldsymbol{i}} + v\hat{\boldsymbol{j}} + w\hat{\boldsymbol{k}}$$

- Acceleration

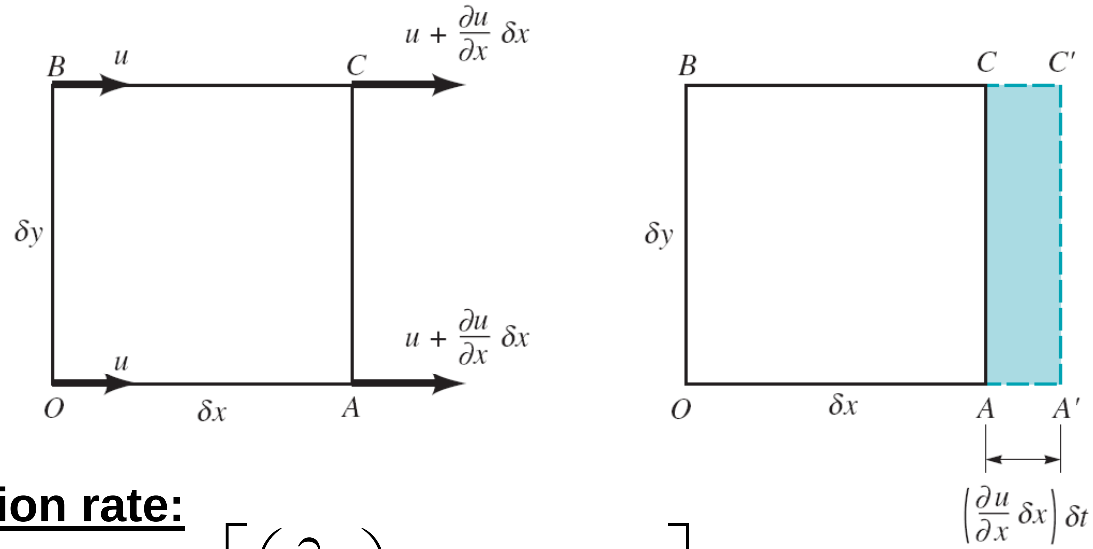
$$\mathbf{a}(r,t) = \frac{\partial \mathbf{V}}{\partial t} + u\frac{\partial \mathbf{V}}{\partial x} + v\frac{\partial \mathbf{V}}{\partial y} + w\frac{\partial \mathbf{V}}{\partial z} = \frac{D\mathbf{V}}{Dt}$$

- Material derivative

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} = \frac{\partial}{\partial t} + (\mathbf{V} \bullet \nabla)$$

Linear motion and deformation

- Let's consider stretching of a fluid element under velocity gradient in one direction

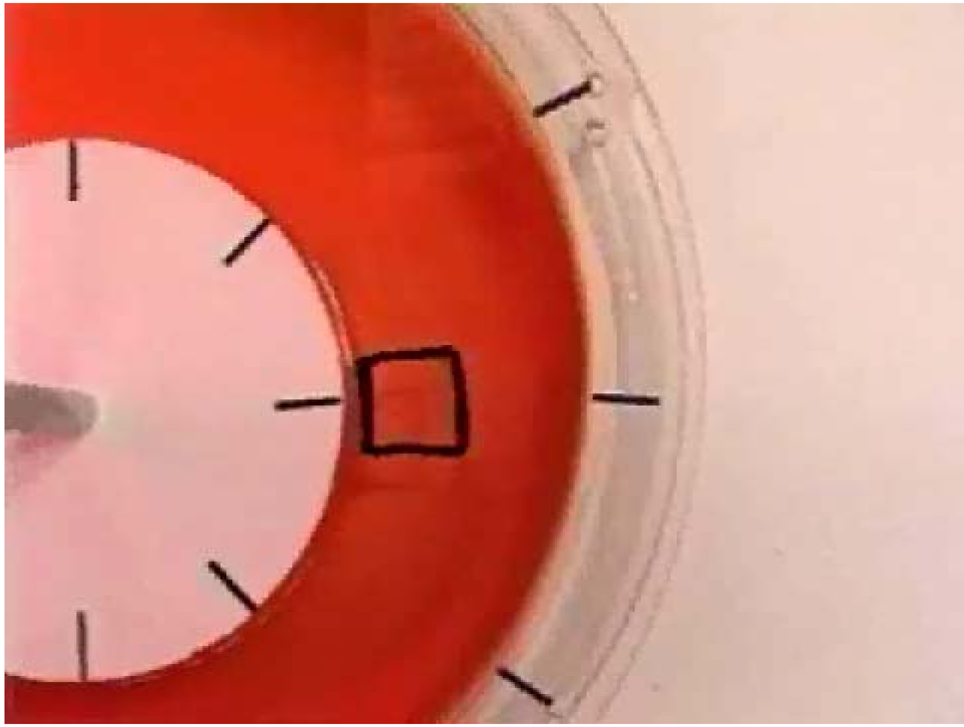


Volumetric dilatation rate:

$$\frac{1}{\delta V}\frac{d(\delta V)}{dt} = \lim_{\delta t \to 0}\left[\frac{\left(\frac{\partial u}{\partial x}\right)\delta x \delta t\, \delta y \delta z}{\delta x \delta y \delta z\, \delta t}\right] = \left(\frac{\partial u}{\partial x}\right)$$

$$\frac{1}{\delta V}\frac{d(\delta V)}{dt} = \left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial v}{\partial y}\right) + \left(\frac{\partial w}{\partial z}\right) = \nabla \cdot \boldsymbol{V}$$

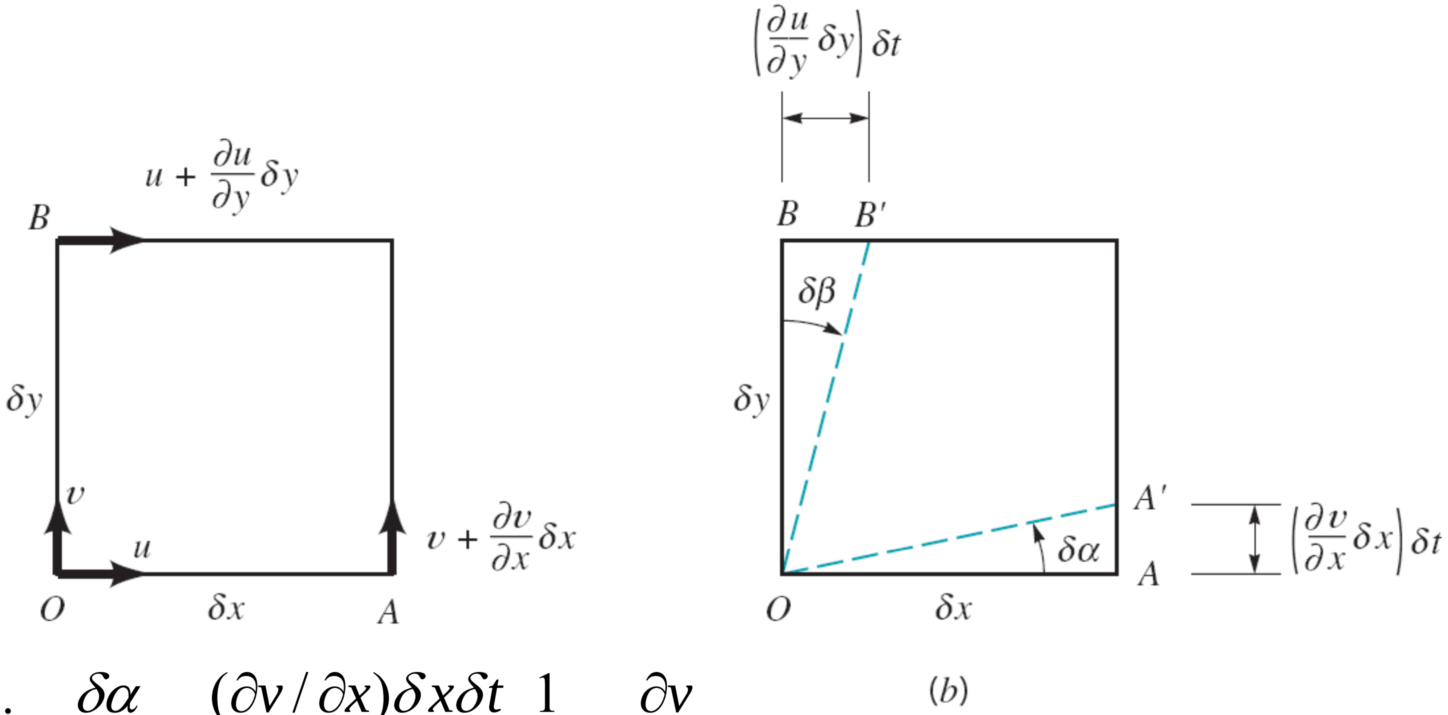
Angular motion and deformation



Fluid elements located in a moving fluid move with the fluid and generally undergo a change in shape (angular deformation).

A small rectangular fluid element is located in the space between concentric cylinders. The inner wall is fixed. As the outer wall moves, the fluid element undergoes an angular deformation. The rate at which the corner angles change (rate of angular deformation) is related to the shear stress causing the deformation

Angular motion and deformation



$$\omega_{OA} = \lim_{\delta t \to 0} \frac{\delta\alpha}{\delta t} \approx \frac{(\partial v / \partial x)\delta x \delta t}{\delta x}\frac{1}{\delta t} = \frac{\partial v}{\partial x}$$

$$\omega_{OB} = \frac{\partial u}{\partial y}$$

- Rotation is defined as the average of those velocities:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Angular motion and deformation

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \qquad \omega_x = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \qquad \omega_y = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$$

$$\boldsymbol{\omega} = \frac{1}{2}\nabla \times \boldsymbol{V} = \frac{1}{2}\begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} =$$

$$= \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{\boldsymbol{i}} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{\boldsymbol{j}} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\boldsymbol{k}}$$

- Vorticity is defined as twice the rotation vector

$$\zeta = 2\omega = \nabla \times \boldsymbol{V}$$

- If rotation (and vorticity) is zero flow is called **irrotational**

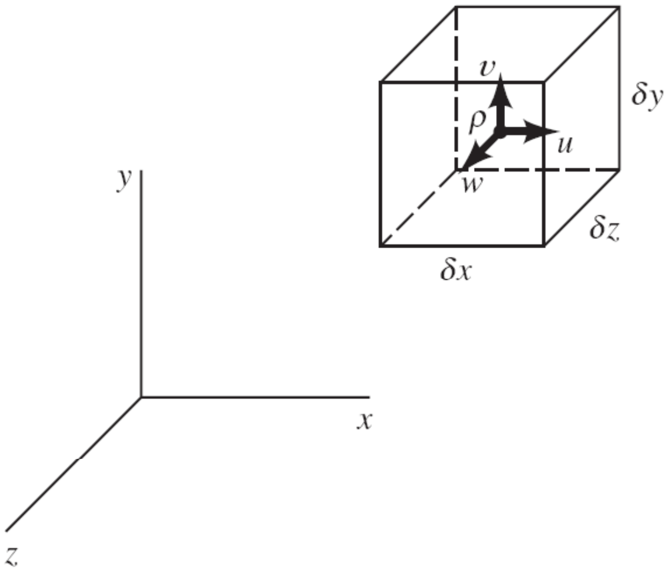
Angular motion and deformation

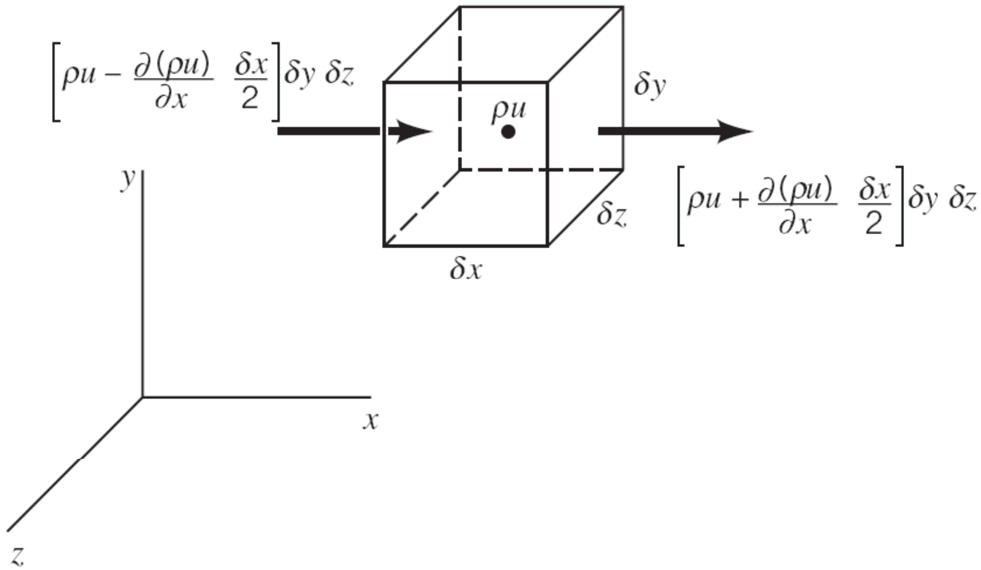
- Rate of shearing strain (or rate of angular deformation) can be defined as sum of fluid element rotations:

$$\dot{\gamma} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$$

Conservation of mass

- As we found before: $$\frac{DM_{Sys}}{Dt} = \frac{\partial}{\partial t}\int_{CV} \rho d\mathcal{V} + \int_{CV} \rho \mathbf{V} \cdot \hat{\mathbf{n}} dA = 0$$





$$\frac{\partial}{\partial t}\int_{CV} \rho d\mathcal{V} = \frac{\partial \rho}{\partial t}\delta x \delta y \delta z$$

Flow of mass in x-direction: $$\frac{\partial(\rho u)}{\partial x}\delta x \delta y \delta z$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0 \qquad \frac{\partial \rho}{\partial t} + \nabla \cdot \rho \mathbf{V} = 0$$

Conservation of mass

- Incompressible flow

$$\nabla \cdot \boldsymbol{V} = 0 \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

- Flow in cylindrical coordinates

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial (r \rho v_r)}{\partial r} + \frac{1}{r}\frac{\partial (\rho v_\theta)}{\partial \theta} + \frac{\partial (\rho v_z)}{\partial z} = 0$$

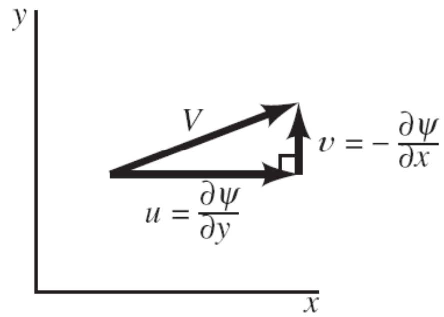
- Incompressible flow in cylindrical coordinates

$$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial (v_\theta)}{\partial \theta} + \frac{\partial (v_z)}{\partial z} = 0$$

Stream function

- 2D incompressible flow

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$



- We can define a scalar function such that

$$u = \frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \psi}{\partial x}$$

Stream function

- Lines along which stream function is const are stream lines:

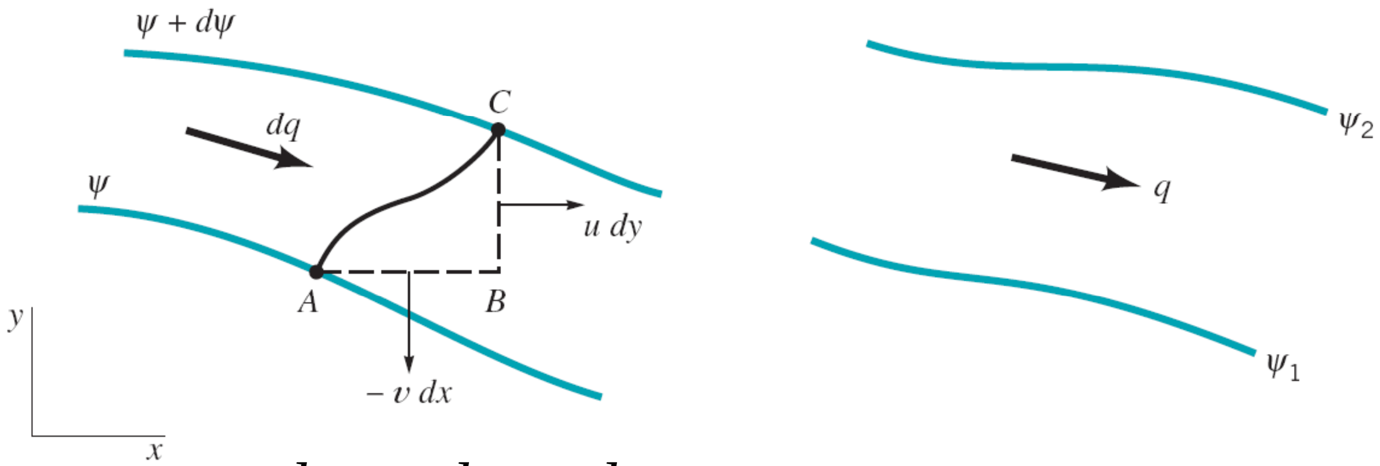
$$\frac{dy}{dx} = \frac{v}{u}$$

Indeed:

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy = -vdx + udy = 0$$

Stream function

- Flow between streamlines



$$dq = udy - vdx$$

(b)

$$dq = \frac{\partial \psi}{\partial y} dy + \frac{\partial \psi}{\partial x} dx = d\psi$$

$$q = \psi_2 - \psi_1$$

Description of forces

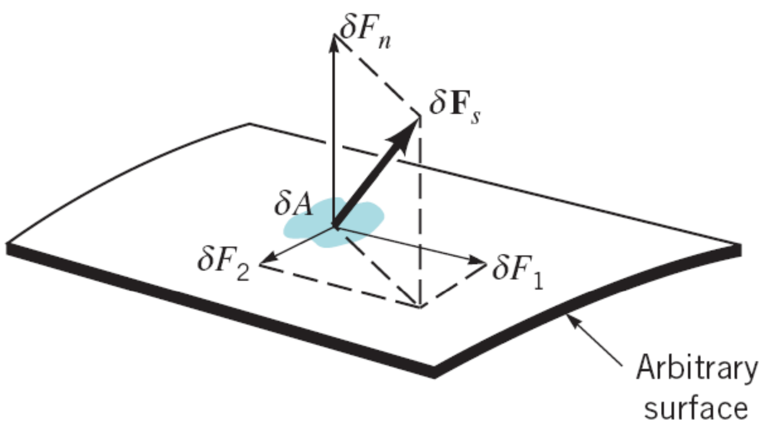
Forces

- **Body forces – distributed through the element, e.g. Gravity**

$$\delta \mathbf{F}_b = \delta m \mathbf{g}$$

- **Surface forces – result of interaction with the surrounding elements: e.g. Stress**

Normal stress



Shearing stresses

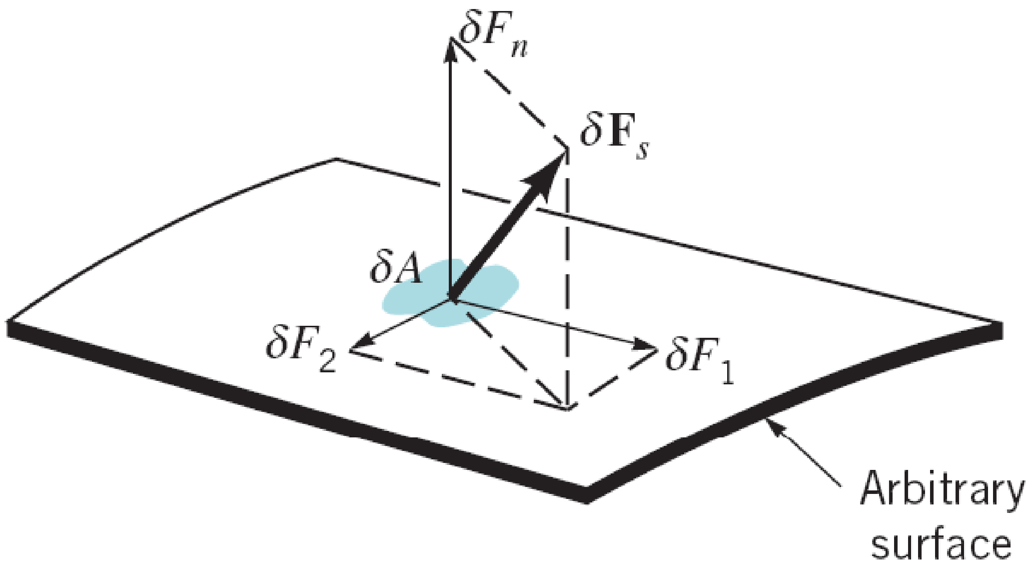
- **Linear forces: Surface tension**

Stress acting on a fluidic element

- normal stress $$\sigma_n = \lim_{\delta A \to 0} \frac{\delta F_n}{\delta A}$$

- shearing stresses

$$\tau_1 = \lim_{\delta A \to 0} \frac{\delta F_1}{\delta A} \qquad \tau_2 = \lim_{\delta A \to 0} \frac{\delta F_2}{\delta A}$$



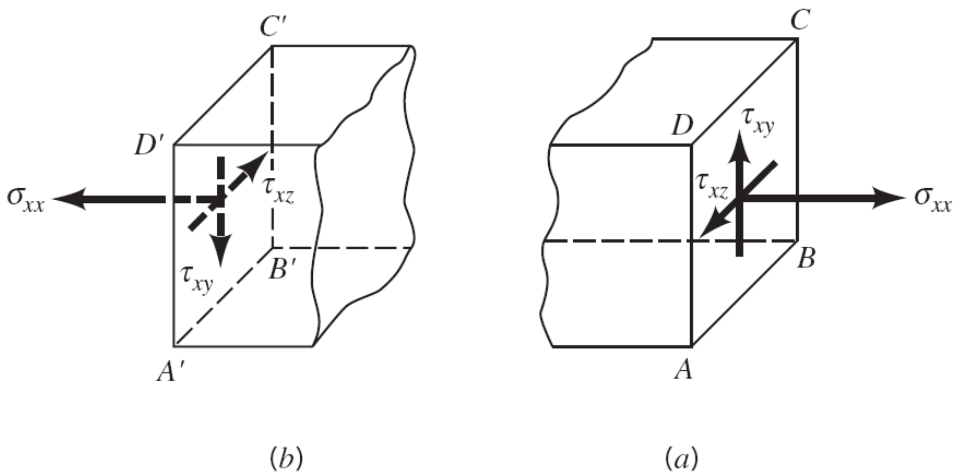
Stresses: double subscript notation

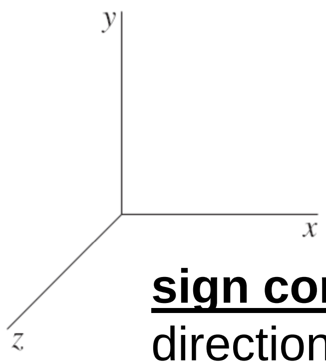
- normal stress: σ_{xx}

- shearing stress: τ_{xy} τ_{xz}

normal to the plane

direction of stress





<u>sign convention</u>: positive stress is directed in positive axis directions if surface normal is pointing in the positive direction

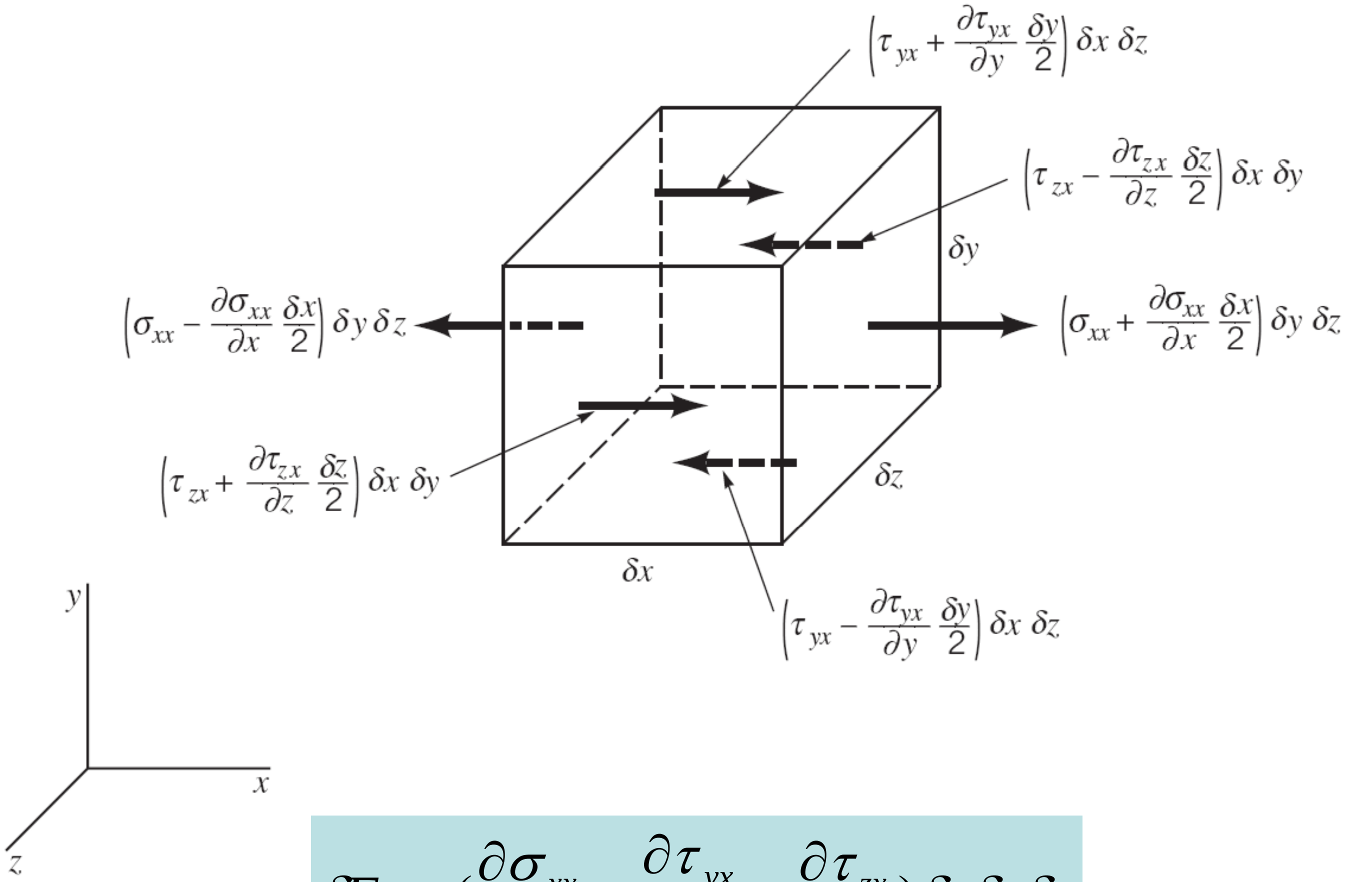
Stress tensor

- To define stress at a point we need to define "stress vector" for all 3 perpendicular planes passing through the point

$$\tau = \begin{pmatrix} \sigma_{11} & \tau_{12} & \tau_{13} \\ \tau_{21} & \sigma_{22} & \tau_{23} \\ \tau_{31} & \tau_{32} & \sigma_{33} \end{pmatrix}$$

Force on a fluid element

- To find force in each direction we need to sum all forces (normal and shearing) acting in the same direction



$$\delta F_x = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right)\delta x \delta y \delta z$$

Differential equation of motion

$$\delta \mathbf{F} = \delta m \mathbf{a}$$

$$\delta F_x = \delta m a_x$$

$$\delta F_y = \delta m a_y, \quad \delta m = \delta x \, \delta y \, \delta z$$

$$\delta F_z = \delta m a_z$$

$$\rho g_x + (\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}) = \rho(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z})$$

Acceleration ("material derivative")

$$\rho g_y + (\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{zy}}{\partial z}) = \rho(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z})$$

$$\rho g_z + (\frac{\partial \sigma_{zz}}{\partial z} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{xz}}{\partial x}) = \rho(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z})$$

Inviscid flow

- no shearing stress in inviscid flow, so

$$-p = \sigma_{xx} = \sigma_{yy} = \sigma_{zz}$$

- equation of motion is reduced to Euler equations

$$\rho g_x - \frac{\partial \sigma_{xx}}{\partial x} = \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right)$$

$$\rho g_y - \frac{\partial \sigma_{yy}}{\partial y} = \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right)$$

$$\rho g_z - \frac{\partial \sigma_{zz}}{\partial x} = \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right)$$

$$\rho \boldsymbol{g} - \nabla p = \rho\left[\frac{\partial \boldsymbol{V}}{\partial t} + (\boldsymbol{V} \cdot \nabla)\boldsymbol{V}\right]$$

Bernoulli equation

- let's write Euler equation for a steady flow along a streamline

$$\rho \boldsymbol{g} - \nabla p = \rho(\boldsymbol{V} \cdot \nabla)\boldsymbol{V}$$

$$\boldsymbol{g} = -g\nabla z$$

$$(\boldsymbol{V} \cdot \nabla)\boldsymbol{V} = \tfrac{1}{2}\nabla(\boldsymbol{V} \cdot \boldsymbol{V}) - \boldsymbol{V} \times (\nabla \times \boldsymbol{V})$$

$$-\rho \boldsymbol{g} \nabla z - \nabla p = \frac{\rho}{2}\nabla(\boldsymbol{V} \cdot \boldsymbol{V}) - \rho \boldsymbol{V} \times (\nabla \times \boldsymbol{V})$$

- now we multiply it by *ds* along the streamline

$$-\rho \boldsymbol{g} \nabla z \cdot d\boldsymbol{s} - \nabla p \cdot d\boldsymbol{s} = \frac{\rho}{2}\nabla(\boldsymbol{V} \cdot \boldsymbol{V}) \cdot d\boldsymbol{s} - \rho \boldsymbol{V} \times (\nabla \times \boldsymbol{V}) \cdot d\boldsymbol{s}$$

$$\frac{dp}{\rho} + \frac{1}{2} d(V^2) + g dz = 0 \qquad or \qquad \frac{p}{\rho} + \frac{V^2}{2} + gz = const$$

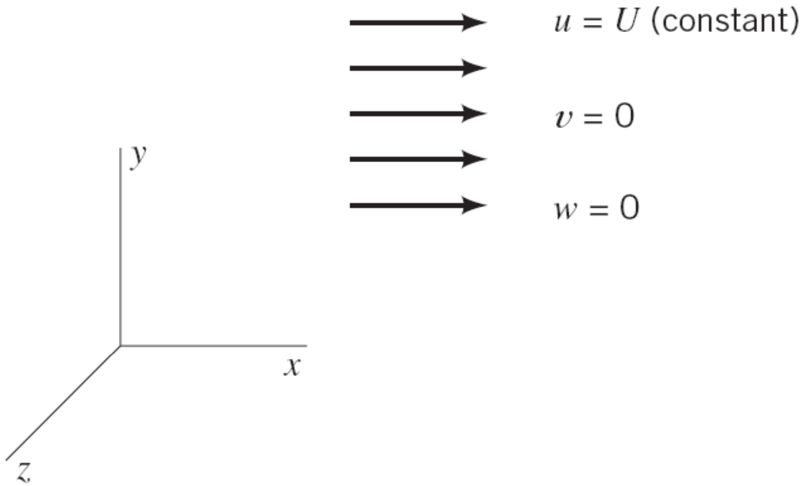
Irrotational Flow

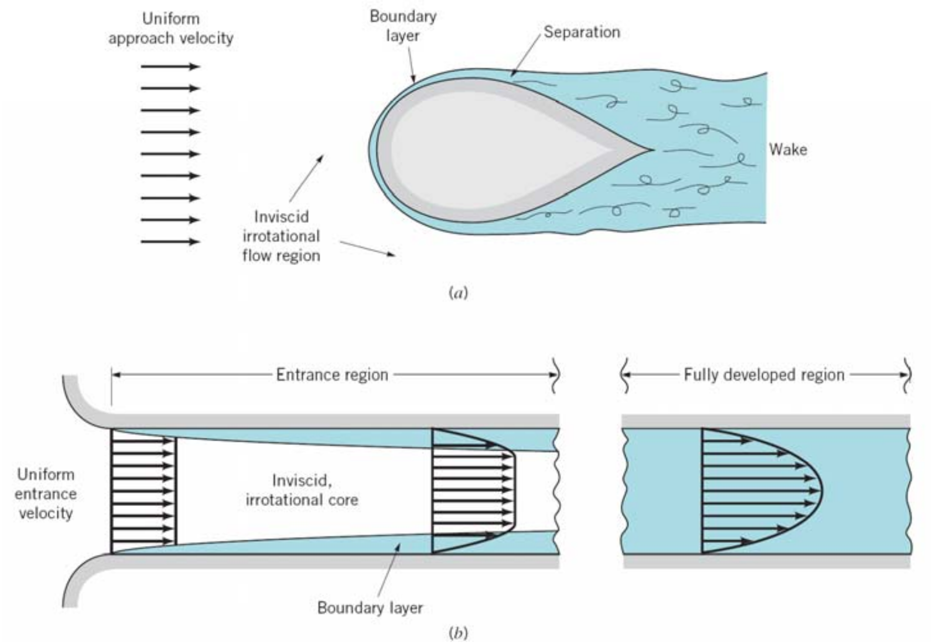
- Analysis of inviscide flow can be further simplified if we assume if the flow is irrotational:

$$\omega_z = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$$\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}; \qquad \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}; \qquad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$$

- Example: uniform flow in x-direction:





Bernoulli equation for irrotational flow

$$-\rho \boldsymbol{g} \nabla z - \nabla p = \frac{\rho}{2} \nabla (\boldsymbol{V} \cdot \boldsymbol{V}) - \rho \boldsymbol{V} \times (\nabla \times \boldsymbol{V})$$

always =0, not only along a stream line

- Thus, Bernoulli equation can be applied between any two points in the flow field

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = const$$

Velocity potential

- equations for irrotational flow will be satisfied automatically if we introduce a scalar function called velocity potential such that:

$$u = \frac{\partial \phi}{\partial x} \qquad v = \frac{\partial \phi}{\partial y} \qquad w = \frac{\partial \phi}{\partial z}$$

$$\boldsymbol{V} = \nabla \phi$$

- This type of flow is called **potential flow**
- As for incompressible flow conservation of mass leads to:

$$\nabla \cdot \boldsymbol{V} = 0, \qquad \nabla^2 \phi = 0$$

Laplace equation

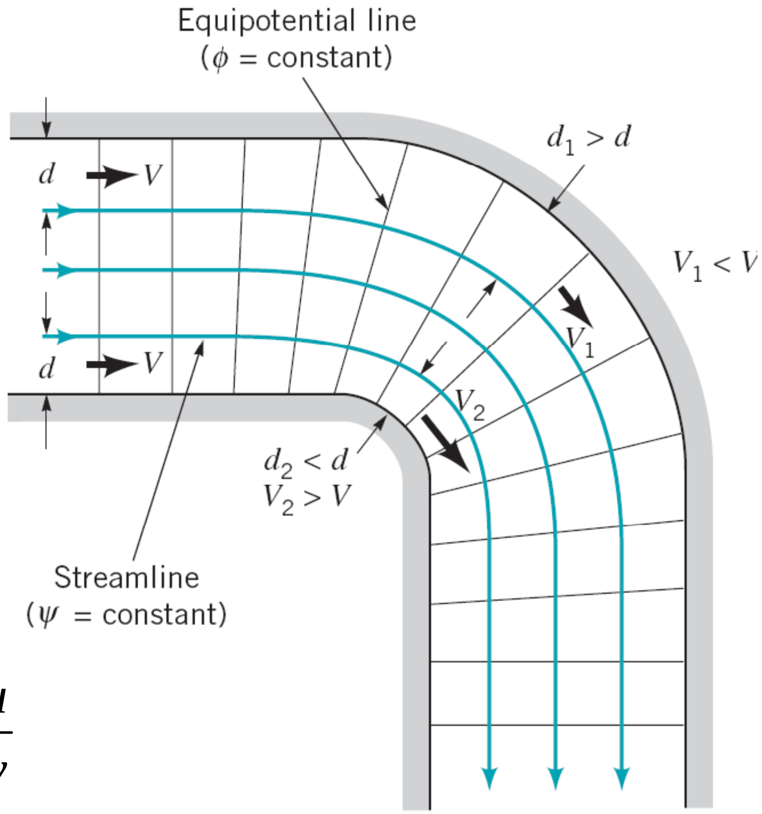
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

- In cylindrical coordinates:

$$\nabla \phi = \frac{\partial \phi}{\partial r}\vec{r} + \frac{1}{r}\frac{\partial \phi}{\partial \theta}\vec{\theta} + \frac{\partial \phi}{\partial z}\vec{z} \qquad \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

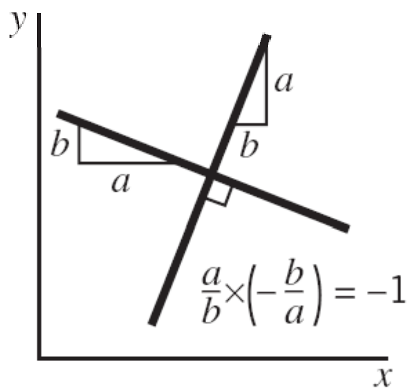
Some basic potential flows

- As Laplace equation is a linear one, the solutions can be added to each other producing another solution;
- stream lines (ψ=const) and equipotential lines (φ=const) are mutually perpendicular



$$\left.\frac{dy}{dx}\right|_{along\ streamline} = \frac{v}{u}$$

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = udx + vdy \Rightarrow \left.\frac{dy}{dx}\right|_{along\ \phi=const} = -\frac{u}{v}$$



<u>Both φ and ψ satisfy Laplace's equation</u>

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \Rightarrow \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right)$$

Uniform flow

- constant velocity, all stream lines are straight and parallel

$$\frac{\partial \phi}{\partial x} = U \qquad \frac{\partial \phi}{\partial y} = 0$$

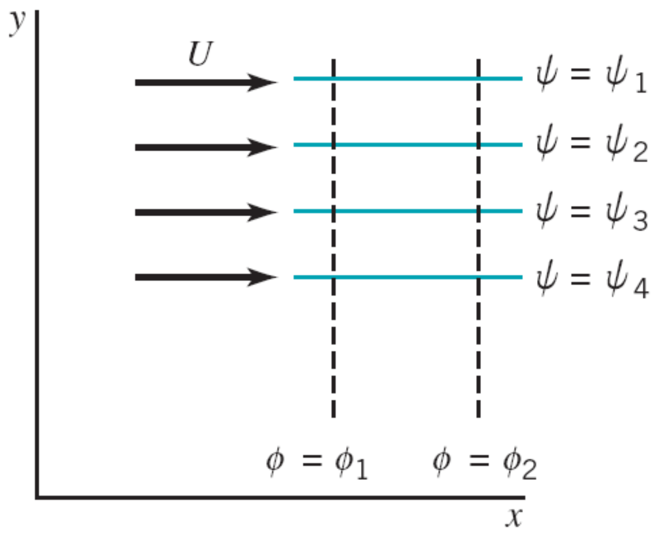
$$\phi = Ux$$

$$\frac{\partial \psi}{\partial y} = U \qquad \frac{\partial \psi}{\partial x} = 0$$

$$\psi = Uy$$

$$\phi = U(x\cos\alpha + y\sin\alpha)$$

$$\psi = U(y\cos\alpha - x\sin\alpha)$$



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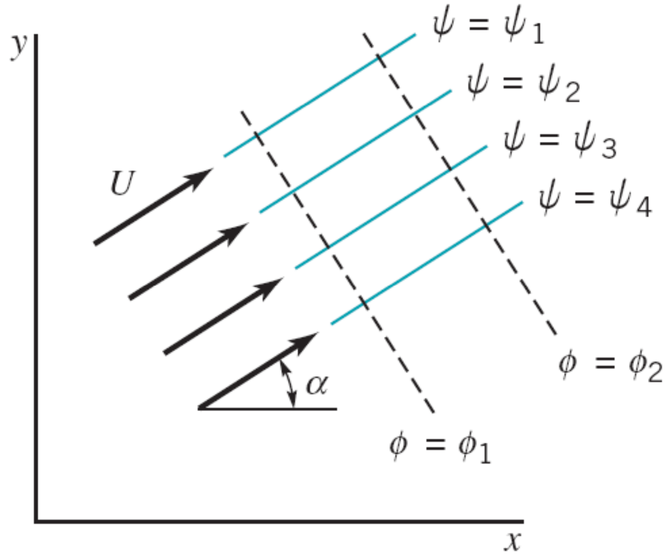


(b)

Source and Sink

- Let's consider fluid flowing radially outward from a line through the origin perpendicular to x-y plane
  from mass conservation:

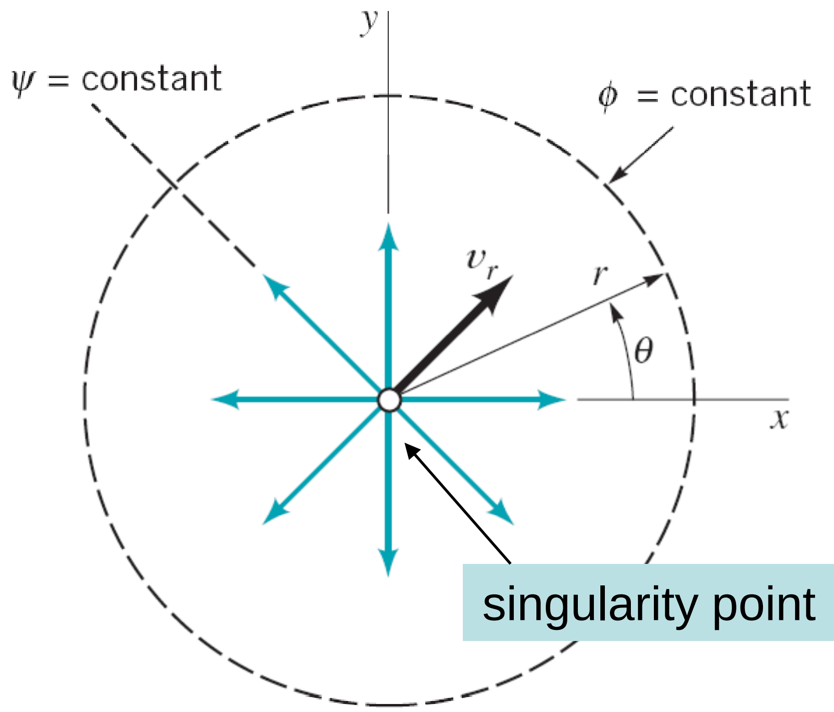
$$(2\pi r)v_r = m$$

$$\frac{\partial \phi}{\partial r} = \frac{m}{2\pi r} \qquad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = 0$$

$$\boxed{\phi = \frac{m}{2\pi}\ln r}$$

$$\frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{m}{2\pi r} \qquad \frac{\partial \psi}{\partial r} = 0$$

$$\boxed{\psi = \frac{m}{2\pi}\theta}$$

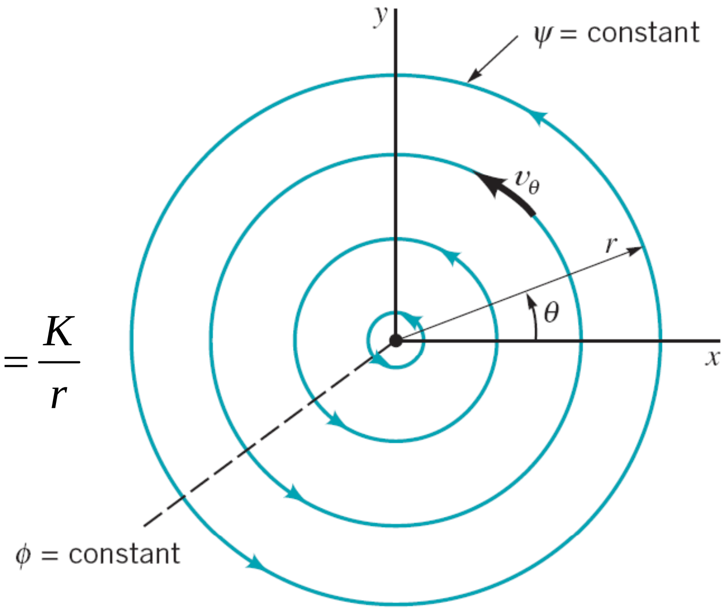


Vortex

- now we consider situation when the stream lines are concentric circles i.e. we interchange potential and stream functions:

$$\phi = K\theta$$
$$\psi = -K \ln r$$

$$v_r = 0$$
$$v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r} = \frac{K}{r}$$



- circulation

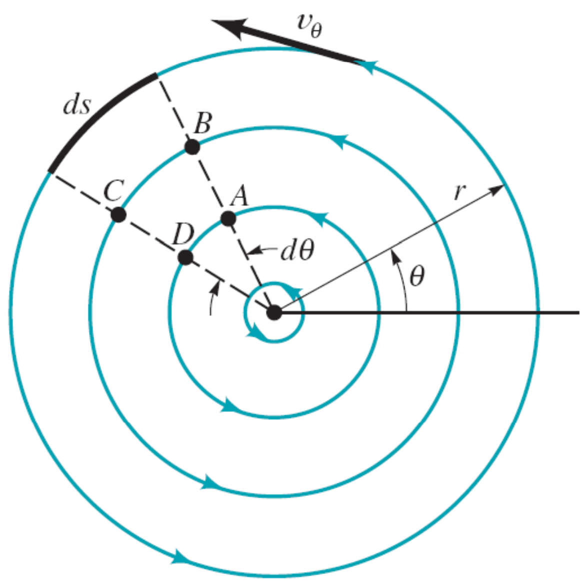
for potential flow

$$\Gamma = \oint_C \mathbf{V} \cdot ds = \oint_C \nabla \phi \cdot ds = \oint_C d\phi = 0$$

- in case of vortex the circulation is zero along any contour except ones enclosing origin

$$\Gamma = \int_0^{2\pi} \frac{K}{r}(r d\theta) = 2\pi K$$

$$\phi = \frac{\Gamma}{2\pi}\theta \qquad \psi = -\frac{\Gamma}{2\pi}\ln r$$



Shape of a free vortex

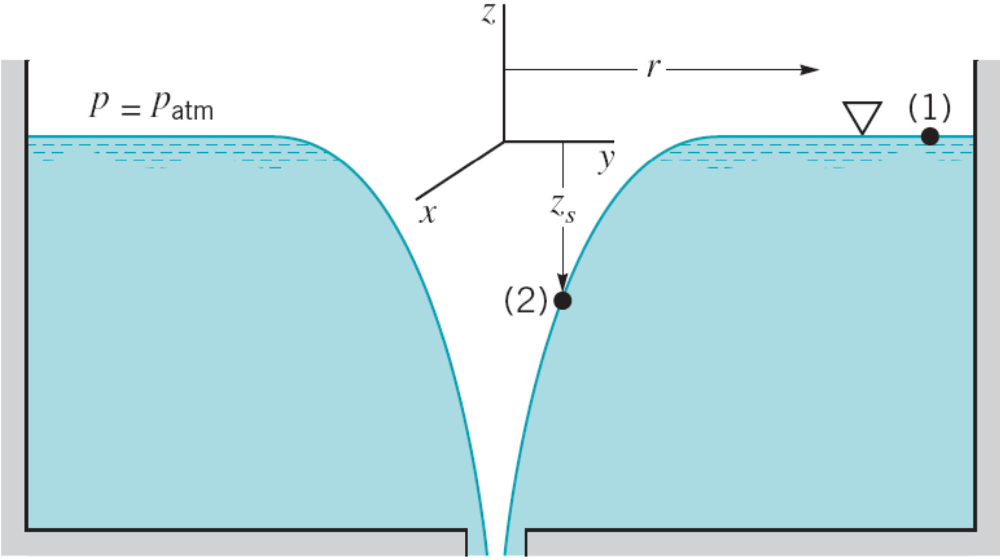
$$\phi = \frac{\Gamma}{2\pi}\theta$$

$$\frac{p}{\rho} + \frac{V^2}{2} + gz = const$$

at the free surface p=0:

$$\frac{V_1^2}{2g} = \frac{V_2^2}{2g} + z$$

$$z = -\frac{\Gamma^2}{8\pi^2 r^2 g}$$

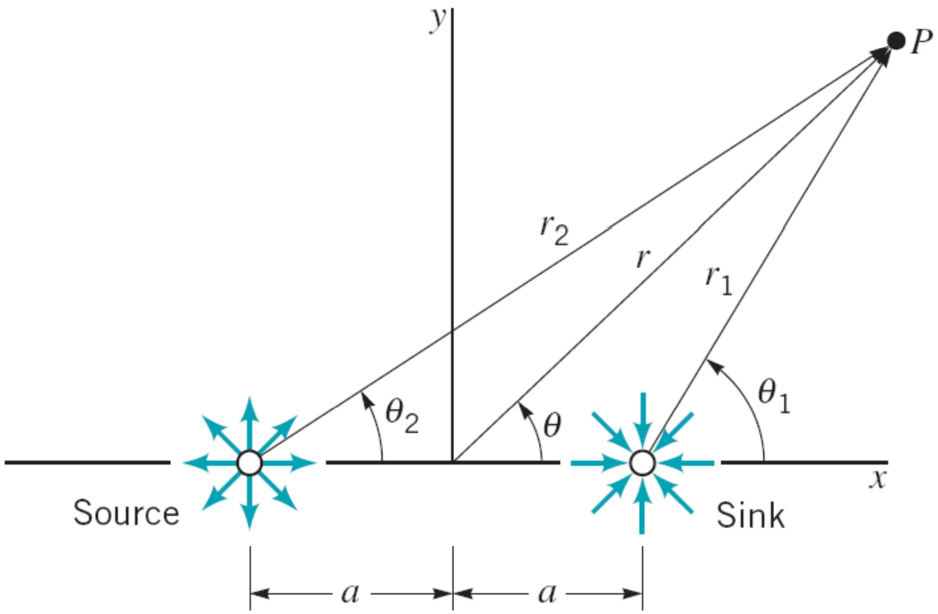


Doublet

- let's consider the equal strength, source-sink pair:

$$\psi = -\frac{m}{2\pi}(\theta_1 - \theta_2)$$

$$\psi = -\frac{m}{2\pi}\tan^{-1}\left(\frac{2ar\sin\theta}{r^2 - a^2}\right)$$

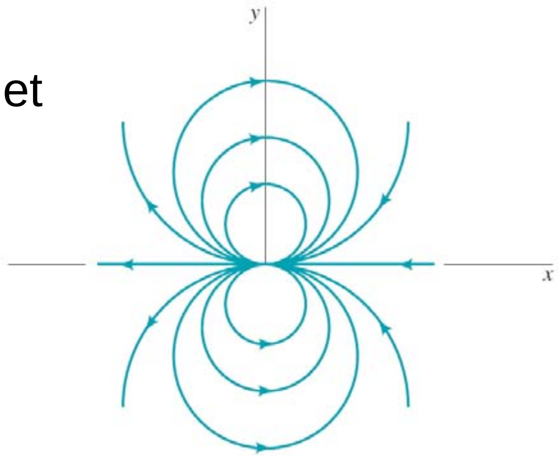


if the source and sink are close to each other:

$$\psi = -\frac{K\sin\theta}{r}$$

$$\phi = \frac{K\cos\theta}{r}$$

K – strength of a doublet



Summary

■ **TABLE 6.1**

Summary of Basic, Plane Potential Flows

Description of Flow Field	Velocity Potential	Stream Function	Velocity Components[a]
Uniform flow at angle α with the x axis (see **Fig. 6.16*b***)	$\phi = U(x \cos \alpha + y \sin \alpha)$	$\psi = U(y \cos \alpha - x \sin \alpha)$	$u = U \cos \alpha$ $v = U \sin \alpha$
Source or sink (see **Fig. 6.17**) $m > 0$ source $m < 0$ sink	$\phi = \dfrac{m}{2\pi} \ln r$	$\psi = \dfrac{m}{2\pi} \theta$	$v_r = \dfrac{m}{2\pi r}$ $v_\theta = 0$
Free vortex (see **Fig. 6.18**) $\Gamma > 0$ counterclockwise motion $\Gamma < 0$ clockwise motion	$\phi = \dfrac{\Gamma}{2\pi} \theta$	$\psi = -\dfrac{\Gamma}{2\pi} \ln r$	$v_r = 0$ $v_\theta = \dfrac{\Gamma}{2\pi r}$
Doublet (see **Fig. 6.23**)	$\phi = \dfrac{K \cos \theta}{r}$	$\psi = -\dfrac{K \sin \theta}{r}$	$v_r = -\dfrac{K \cos \theta}{r^2}$ $v_\theta = -\dfrac{K \sin \theta}{r^2}$

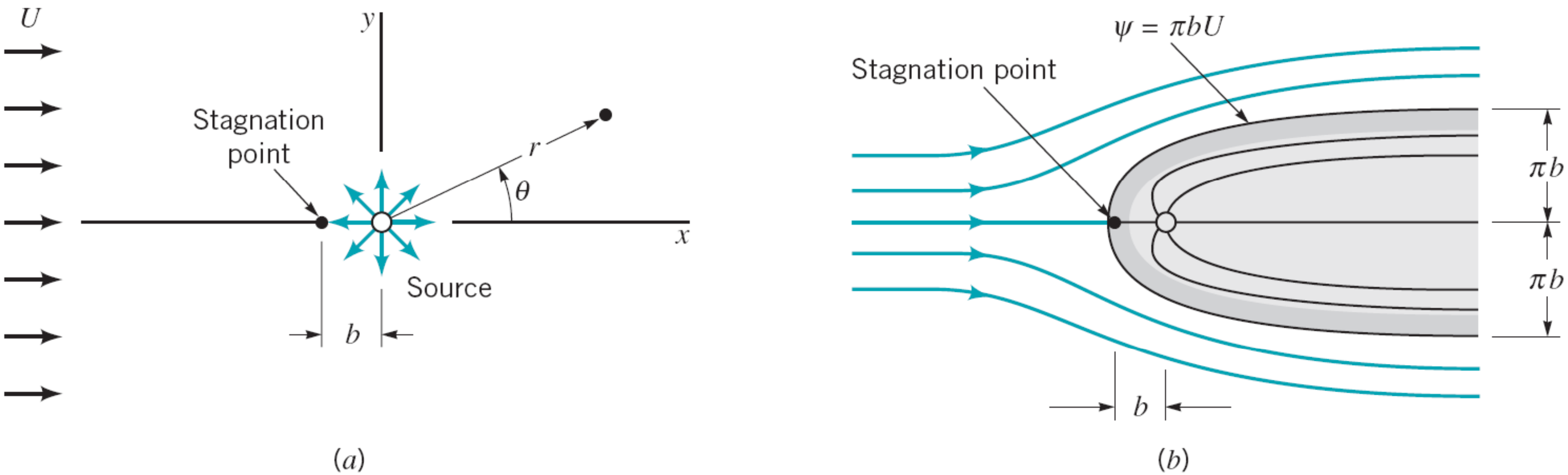
[a]Velocity components are related to the velocity potential and stream function through the relationships:

$$u = \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \qquad v = \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \qquad v_r = \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \qquad v_\theta = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}.$$

Superposition of basic flows

- basic potential flows can be combined to form new potentials and stream functions. This technique is called the **method of superposition**

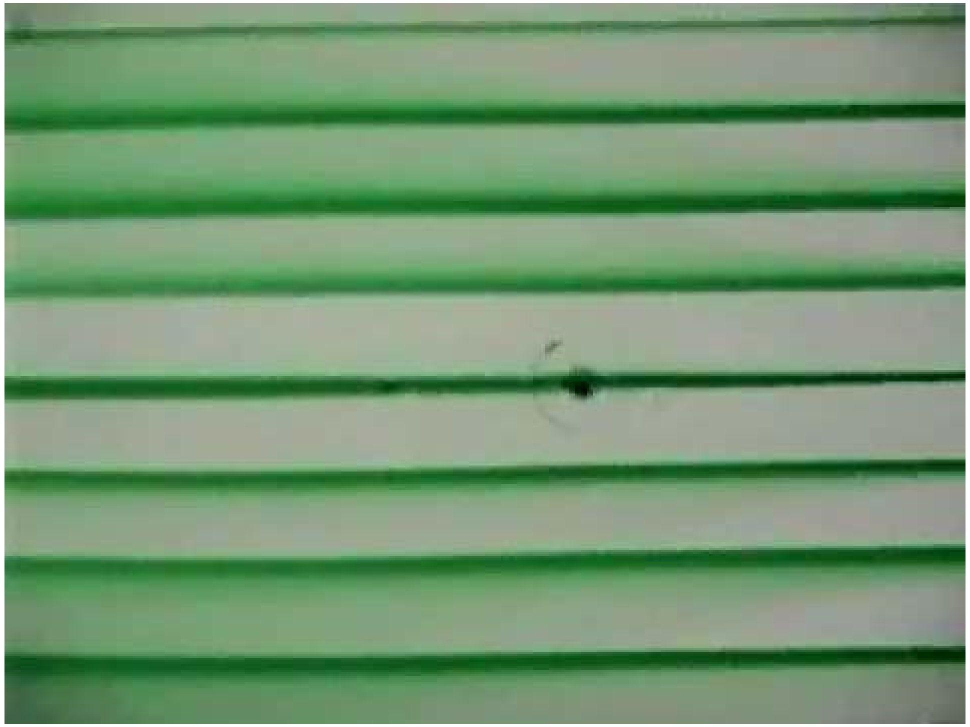
- superposition of source and uniform flow



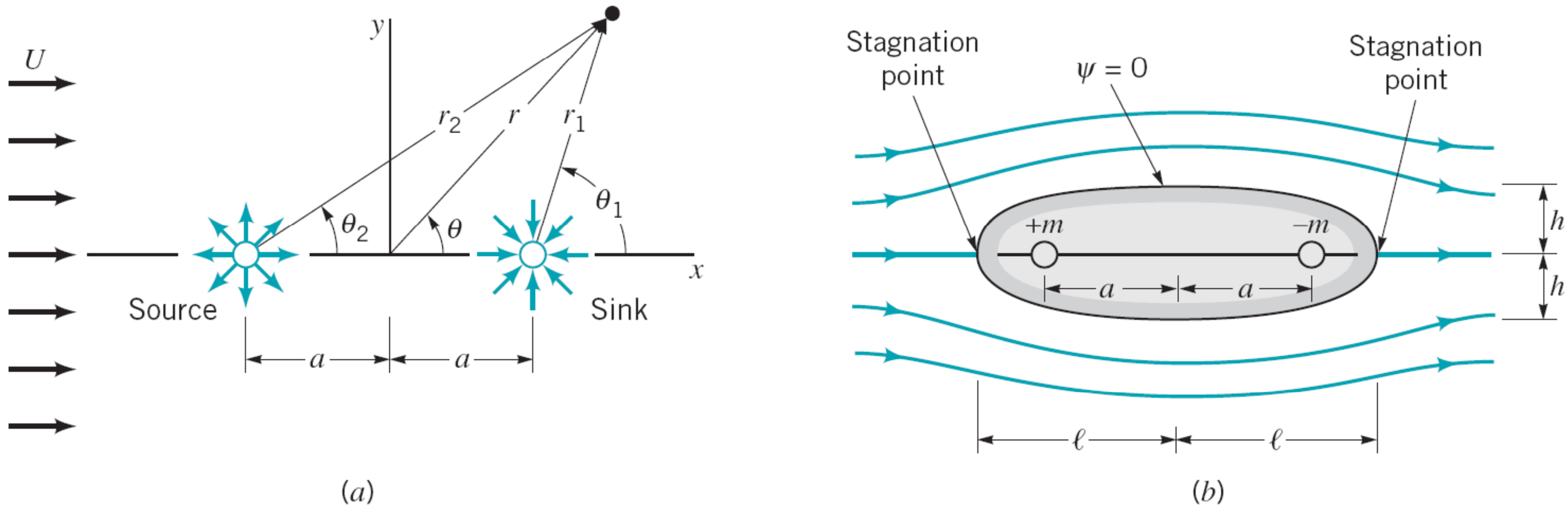
$$\psi = Ur\sin\theta + \frac{m}{2\pi}\theta \qquad \phi = Ur\cos\theta + \frac{m}{2\pi}\ln r$$

Superposition of basic flows

- Streamlines created by injecting dye in steadily flowing water show a uniform flow. Source flow is created by injecting water through a small hole. It is observed that for this combination the streamline passing through the stagnation point could be replaced by a solid boundary which resembles a streamlined body in a uniform flow. The body is open at the downstream end and is thus called a halfbody.



Rankine Ovals



- a closed body can be modeled as a combination of a uniform flow and source and a sink of equal strength

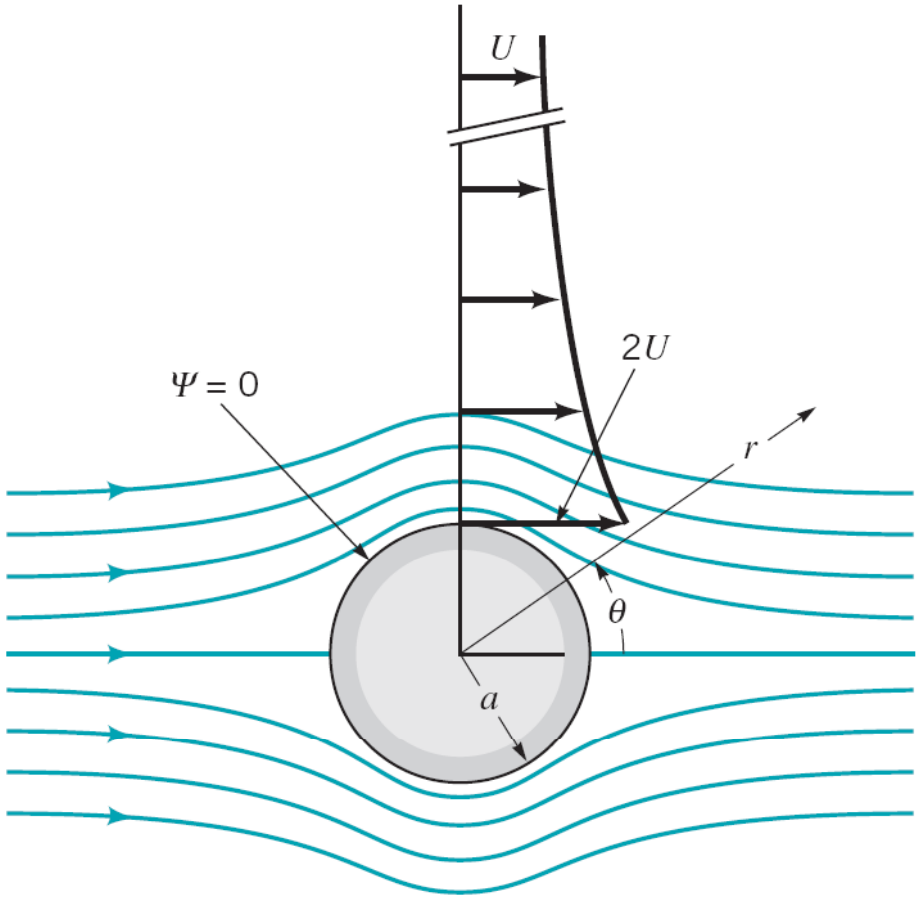
$$\psi = Ur\sin\theta - \frac{m}{2\pi}(\theta_1 - \theta_2) \qquad \phi = Ur\cos\theta - \frac{m}{2\pi}(\ln r_1 - \ln r_2)$$

Flow around circular cylinder

- when the distance between source and sink approaches 0, shape of Rankine oval approaches a circular shape

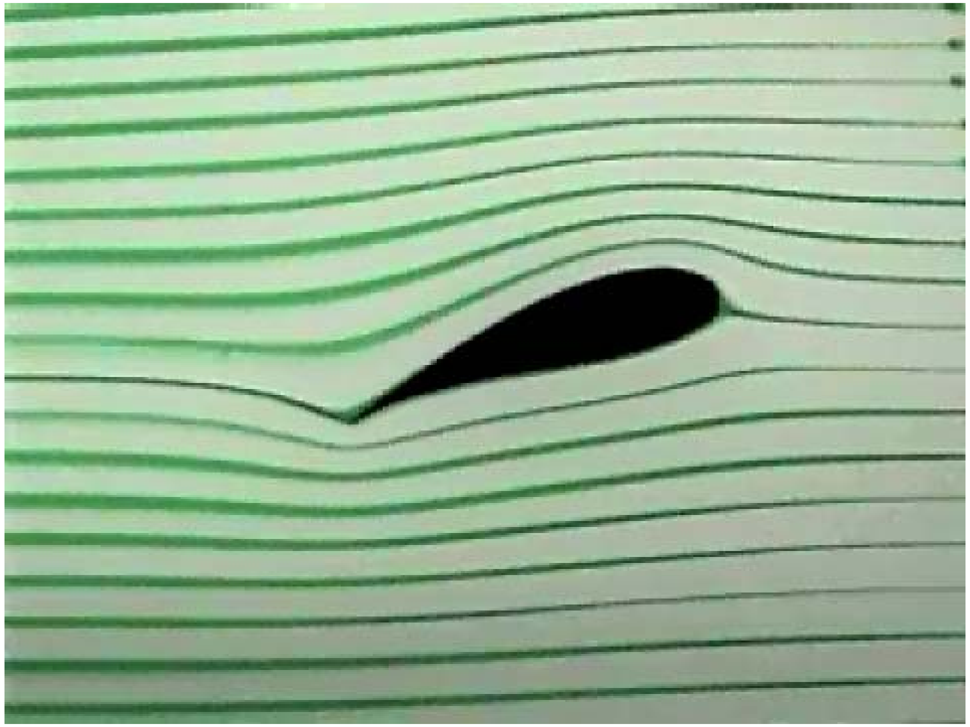
$$\psi = Ur\sin\theta - \frac{K\sin\theta}{r}$$

$$\phi = Ur\cos\theta + \frac{K\cos\theta}{r}$$



Potential flows

- Flow fields for which an incompressible fluid is assumed to be frictionless and the motion to be irrotational are commonly referred to as potential flows.
- Paradoxically, potential flows can be simulated by a slowly moving, viscous flow between closely spaced parallel plates. For such a system, dye injected upstream reveals an approximate potential flow pattern around a streamlined airfoil shape. Similarly, the potential flow pattern around a bluff body is shown. Even at the rear of the bluff body the streamlines closely follow the body shape. Generally, however, the flow would separate at the rear of the body, an important phenomenon not accounted for with potential theory.



Viscous Flow

- Moving fluid develops additional components of stress due to viscosity. For incompressible fluids:

$$\sigma_{xx} = -p + 2\mu \frac{du}{dx}$$

$$\sigma_{yy} = -p + 2\mu \frac{dv}{dy}$$

$$\sigma_{zz} = -p + 2\mu \frac{dw}{dz}$$

$$\tau_{xy} = \tau_{yx} = \mu(\frac{du}{dy} + \frac{dv}{dx})$$

$$\tau_{yz} = \tau_{zy} = \mu(\frac{dv}{dz} + \frac{dw}{dy})$$

$$\tau_{zx} = \tau_{xz} = \mu(\frac{du}{dz} + \frac{dw}{dx})$$

for viscous flow normal stresses are not necessary the same in all directions

Navier-Stokes Equations

$$\rho(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z})=-\frac{\partial p}{\partial x}+\rho g_x+\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})$$

$$\rho(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z})=-\frac{\partial p}{\partial y}+\rho g_y+\mu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2})$$

$$\rho(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z})=-\frac{\partial p}{\partial z}+\rho g_z+\mu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

- 4 equations for 4 unknowns (*u*,*v*,*w*,*p*)
- Analytical solution are known for only few cases

General form of the Navier-Stokes Equation

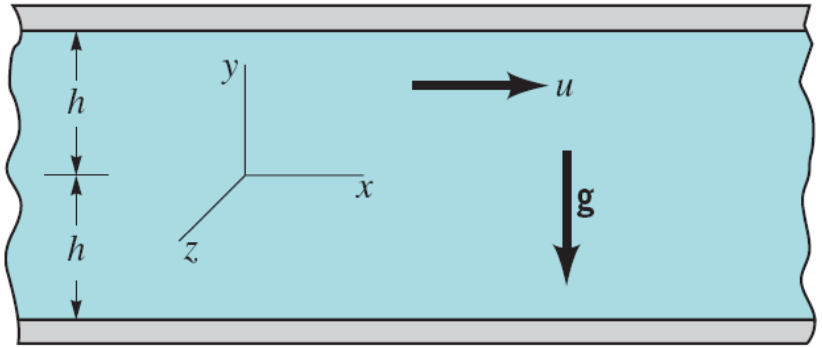
$$\tau_{ij} = -\left(p + \frac{2}{3}\mu \nabla \cdot \vec{V} \right)\delta_{ij} + 2\mu e_{ij}$$

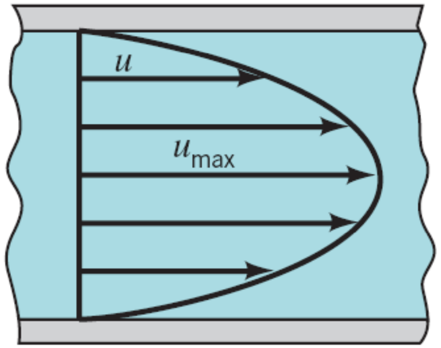
$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu \left(\nabla^2 \vec{V} + \frac{1}{3}\frac{\partial^2}{\partial x_i} \nabla \cdot \vec{V} \right)$$

- incompressible fluid

$$\rho \frac{D\vec{V}}{Dt} = -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{V}$$

Steady Laminar Flow between parallel plates





$$\rho(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z})=-\frac{\partial p}{\partial x}+\rho g_x+\mu(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2})$$

$$\rho(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z})=-\frac{\partial p}{\partial y}+\rho g_y+\mu(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2})$$

$$\rho(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z})=-\frac{\partial p}{\partial z}+\rho g_z+\mu(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2})$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

$$v=0; w=0 \Rightarrow \frac{\partial u}{\partial x}=0$$

$$0=-\frac{\partial p}{\partial x}+\mu(\frac{\partial^2 u}{\partial y^2})$$

$$0=-\frac{\partial p}{\partial y}-\rho g$$

$$0=-\frac{\partial p}{\partial z}$$

$$v = 0; w = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$$

$$0 = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial y^2}\right)$$

$$0 = -\frac{\partial p}{\partial y} - \rho g$$

$$0 = -\frac{\partial p}{\partial z}$$

$$v = 0; w = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$$

$$\frac{du}{dy} = \frac{1}{\mu}\frac{\partial p}{\partial x} y + C_1 \Rightarrow u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + C_1 y + C_2$$

$$p = -\rho g y + f(x)$$

Boundary condition (no slip) $u(h) = u(-h) = 0$

Velocity profile $$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - h^2)$$

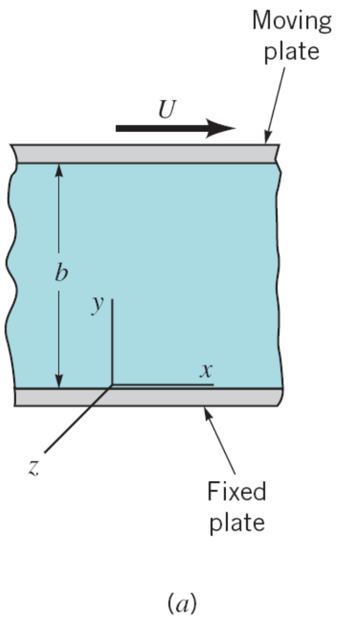
Flow rate $$q = \int_{-h}^{h} u\,dy = \int_{-h}^{-h} \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - h^2)\,dy = -\frac{2h^3}{3\mu}\left(\frac{\partial p}{\partial x}\right)$$

What is maximum velocity (u_{max}) and average velocity?

No-slip boundary condition

- Boundary conditions are needed to solve the differential equations governing fluid motion. One condition is that any viscous fluid sticks to any solid surface that it touches.
- Clearly a very viscous fluid sticks to a solid surface as illustrated by pulling a knife out of a jar of honey. The honey can be removed from the jar because it sticks to the knife. This no-slip boundary condition is equally valid for small viscosity fluids. Water flowing past the same knife also sticks to it. This is shown by the fact that the dye on the knife surface remains there as the water flows past the knife.

Couette flow



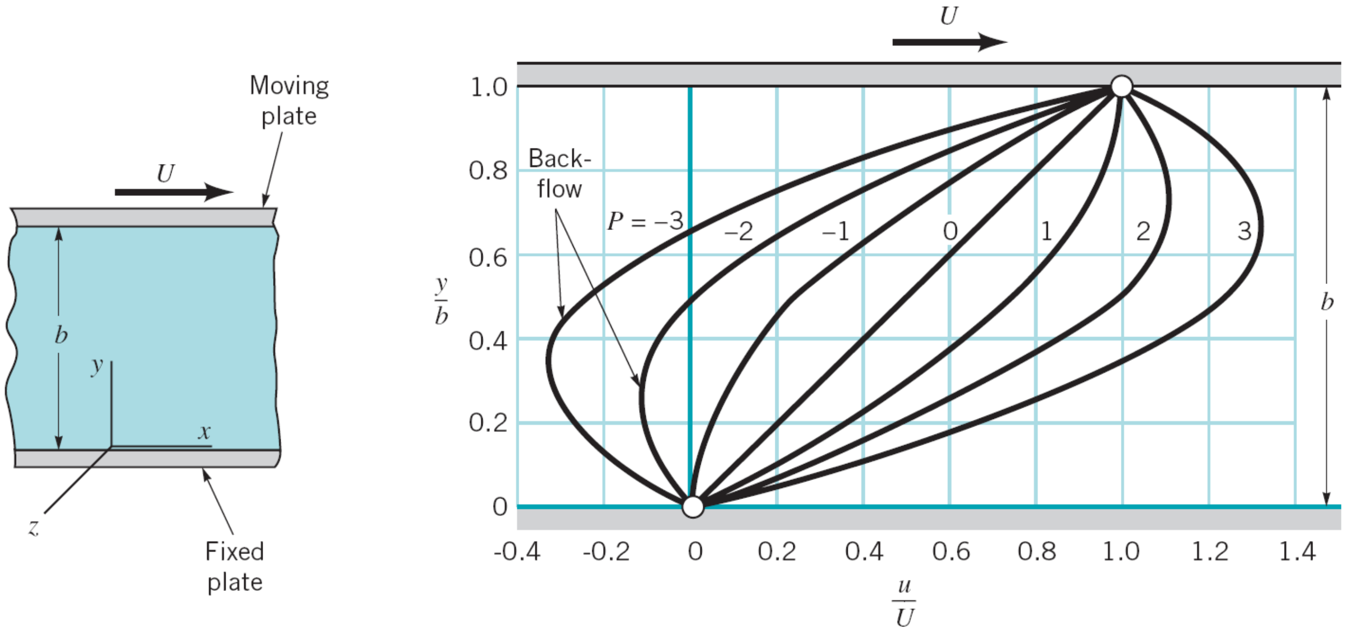
$$v = 0; w = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$$

$$\frac{du}{dy} = \frac{1}{\mu}\frac{\partial p}{\partial x} y + C_1 \Rightarrow u = \frac{1}{2\mu}\frac{\partial p}{\partial x} y^2 + C_1 y + C_2$$

$$p = -\rho g y + f(x)$$

Boundary condition (no slip) $u(0) = 0; u(b) = U$

Please find velocity profile and flow rate



(a) (b)

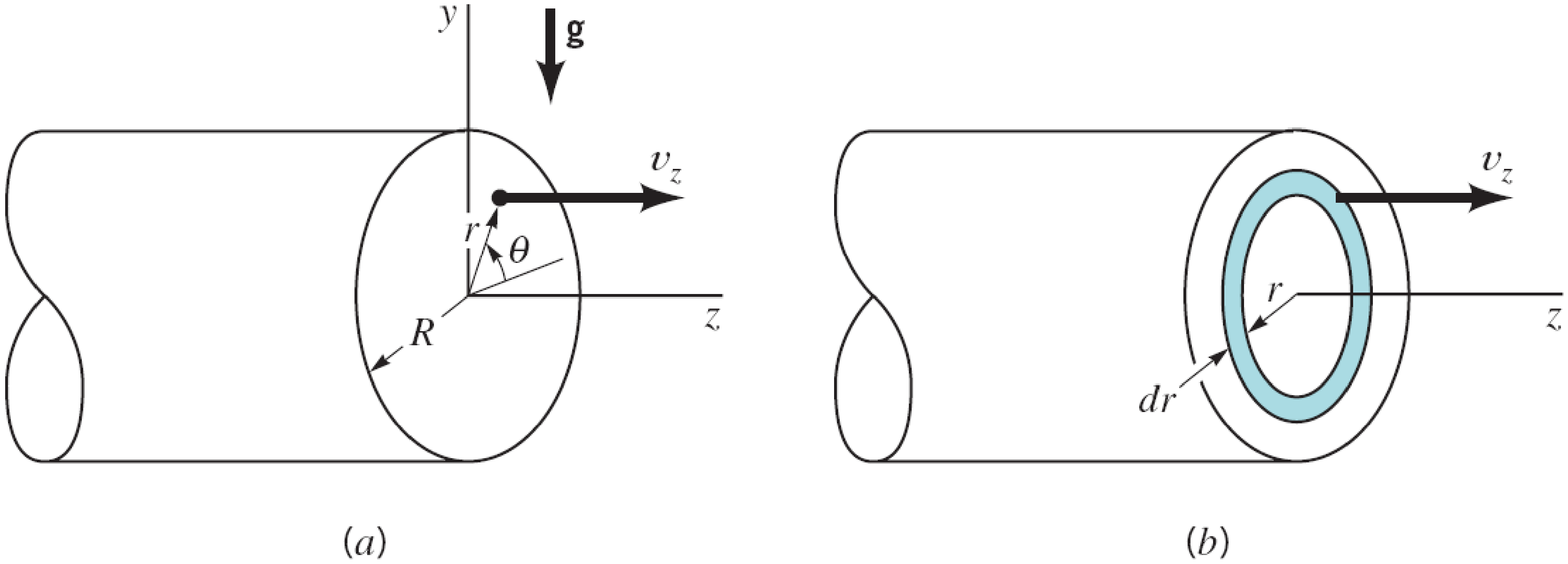
Boundary condition (no slip) $u(0)=0; u(b)=U$

Velocity profile $u = U\frac{y}{b} + \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - by)$

$$\frac{u}{U} = \frac{y}{b} - \underbrace{\frac{b^2}{2\mu U}\frac{\partial p}{\partial x}}(\frac{y}{b})(1-\frac{y}{b})$$

Dimensionless parameter P

Hagen-Poiseuille flow



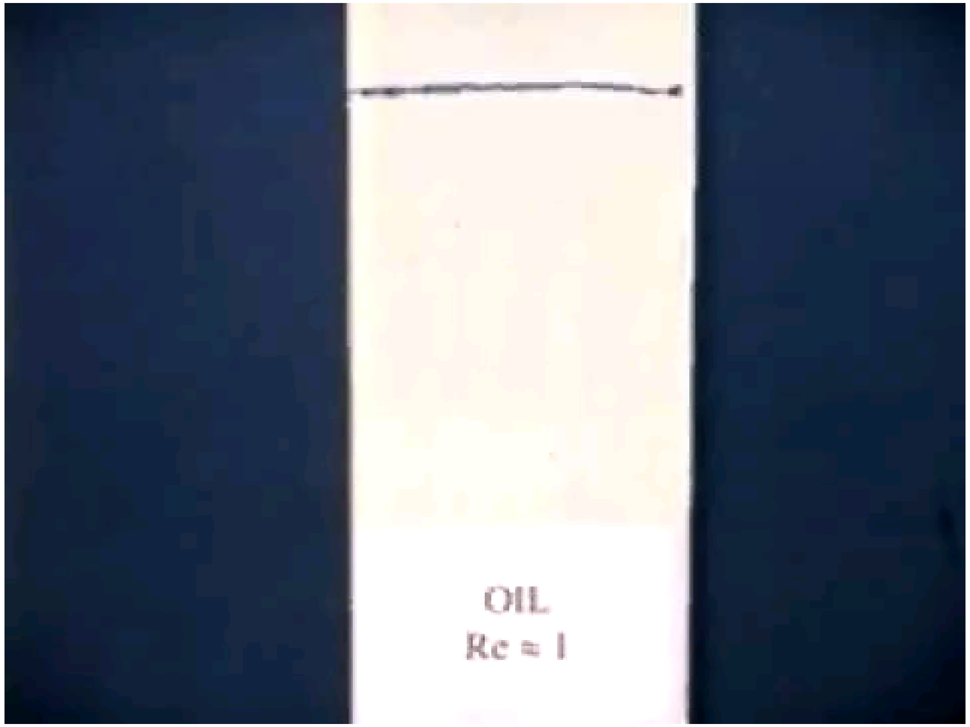
Poiseuille law:

$$Q = \frac{\pi R^4 \Delta p}{8 \mu l};$$

$$v = \frac{R^2 \Delta p}{8 \mu l}$$

Laminar flow

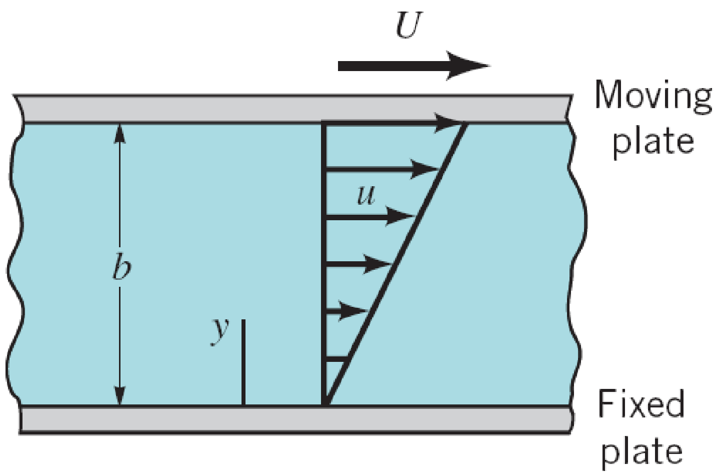
- The velocity distribution is parabolic for steady, laminar flow in circular tubes. A filament of dye is placed across a circular tube containing a very viscous liquid which is initially at rest. With the opening of a valve at the bottom of the tube the liquid starts to flow, and the parabolic velocity distribution is revealed. Although the flow is actually unsteady, it is quasi-steady since it is only slowly changing. Thus, at any instant in time the velocity distribution corresponds to the characteristic steady-flow parabolic distribution.



Problems

- **6.74** Oil SAE30 at 15.6C steadily flows between fixed horizontal parallel plates. The pressure drop per unit length is 20kPa/m and the distance between the plates is 4mm, the flow is laminar.
  Determine the volume rate of flow per unit width; magnitude and direction of the shearing stress on the bottom plate; velocity along the centerline of the channel

- **6.8** An incompressible viscous fluid is placed between two large parallel plates. The bottom plate is fixed and the top moves with the velocity U. Determine:
  - volumetric dilation rate;
  - rotation vector;
  - vorticity;
  - rate of angular deformation.



$$u = U\frac{y}{b}$$