Lecture 5

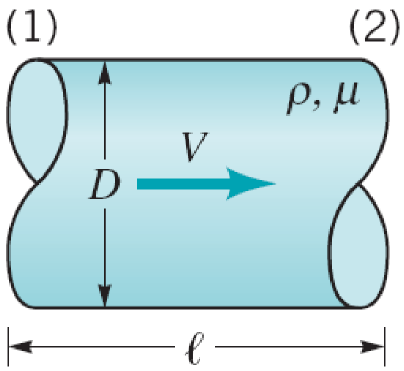
Similitude and Modeling

Simultitude and Dimensional Analysis

- Although many problems in fluid mechanics can be solved analytically, still in most of the situations require combination of analysis and experiment
- ***<u>Similitude</u>*** – a concept aimed on **defining similarity** between different system and finding how the data obtained on a model system can be transferred to other systems

Dimensional Analysis

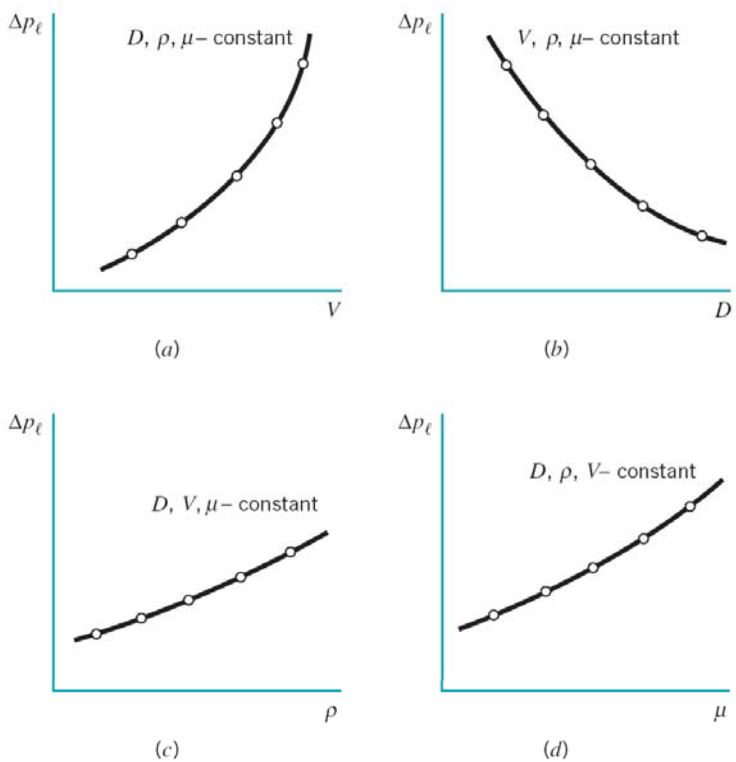
- Let’s consider experiment involving flow through a pipe



$$\Delta p_\ell = (p_1 - p_2)/\ell$$

$$\Delta p_l = f(D, \rho, \mu, V)$$

Straightforward approach: all variables should be varied

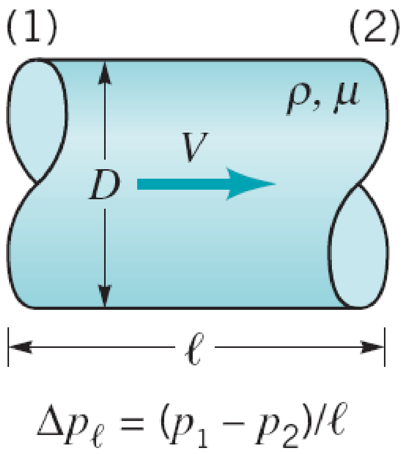


Dimensional Analysis

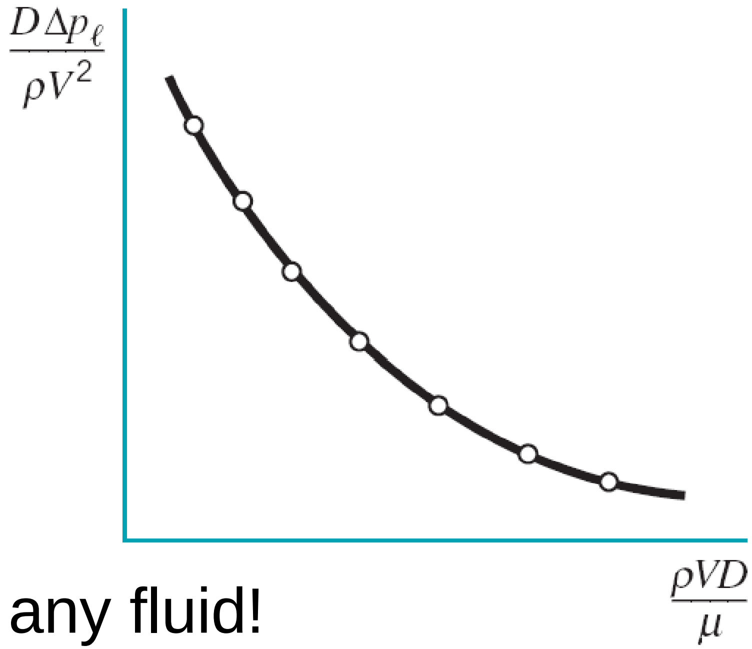
- as you will learn in this lecture, it’s sufficient to make one plot:

Dimensionless groups approach

$$\frac{D\Delta p_l}{pV^2} = \Phi(\frac{\rho VD}{\mu})$$



$$\Delta p_l = f(D, \rho, \mu, V)$$



- and it’s valid for any pipe size and any fluid!

Buckingham Pi Theorem

- If an equation involving ***k*** variables is dimensionally homogeneous, it can be reduced to a relationship among ***k-r*** independent dimensionless products, where ***r*** is the minimum number of reference dimensions required to describe the variables

$$X_1 = f(X_2, X_3, \ldots, X_k) \quad \Rightarrow \quad \Pi_1 = \Phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$$

Pi-terms

- reference dimensions will be usually 3 basic dimensions ***M, L, T*** or ***F, L, T***, some times only two of them or even one might be required.

Buckingham Pi-Theorem

- How to go from the original variables to Pi terms?

- Several systematic methods exists, one of them is the **method of repeating variables**

Method of repeating variables

- Step 1: List all variables involved in the problem
  - incl. geometry, fluid property, external factors (pressure) etc.
- Step 2: Express each variable in terms of basic dimensions. For a typical problem M, L, T or F, L, T ($F=ma \Rightarrow F=MLT^{-2}$)
- Step 3: Determine the required number of Pi terms
  - Buckingham theorem: ***k-r***
- Step 4: Select the number of repeating variables:
  - equal to the number of basic dimensions,
  - dimensionally independent,
  - covering all dimensions

Method of repeating variables

- Step 5: Form a Pi term by multiplying one of the nonrepeating variables by the product of the repeating variables, each raised to an exponent that will make the combination dimensionless

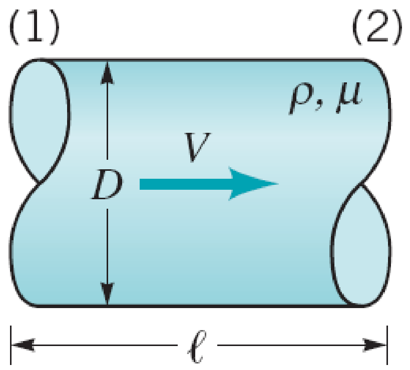
$$\Pi_i = x_i x_1^{\alpha} x_3^{\beta} x_3^{\gamma}$$

non-repeating variable

repeating variables

- Step 6: Repeat Step 5 for each remaining nonrepeating variable
- Step 7: Check all resulting Pi terms to ensure they are dimensionless
- Step 8: Express the final form as a relationship among Pi terms and think about what it means

$$\Pi_1 = \Phi(\Pi_2, \Pi_3, \ldots, \Pi_{k-r})$$



$$\Delta p_\ell = (p_1 - p_2)/\ell$$

$$\Delta p_l \doteq FL^{-3}$$
$$D \doteq L$$
$$\rho \doteq FL^{-4}T^{2}$$
$$\mu \doteq FL^{-2}T$$
$$V \doteq LT^{-1}$$

$$\Delta p_l = f(D, \rho, \mu, V)$$

- 5 variables
- 3 reference dimensions
- therefore 2 Pi terms required
- dependent variable (here, Δp_l) is never

chosen as a repeating.

$$\Pi_1 = \Delta p_l D^a V^b \rho^c$$
$$(FL^{-3})(L)^a(LT^{-1})^b(FL^{-4}T^2)^c \doteq F^0L^0T^0$$

$$F: 1 + c = 0$$
$$L: -3 + a + b - 4c = 0$$
$$T: -b + 2c = 0$$

$$a = 1$$
$$b = -2$$
$$c = -1$$

$$\Pi_1 = \Delta p_l D^1 V^{-2} \rho^{-1} = \frac{\Delta p_l D}{\rho V^2}$$

$$\Delta p_l \doteq FL^{-3}$$
$$D \doteq L$$
$$\rho \doteq FL^{-4}T^{2}$$
$$\mu \doteq FL^{-2}T$$
$$V \doteq LT^{-1}$$

$$\Pi_2 = \mu D^a V^b \rho^c$$
$$(FL^{-2}T)(L)^a(LT^{-1})^b(FL^{-4}T^2)^c \doteq F^0L^0T^0$$

⇨

$$F: 1 + c = 0$$
$$L: -2 + a + b - 4c = 0$$
$$T: 1 - b + 2c = 0$$

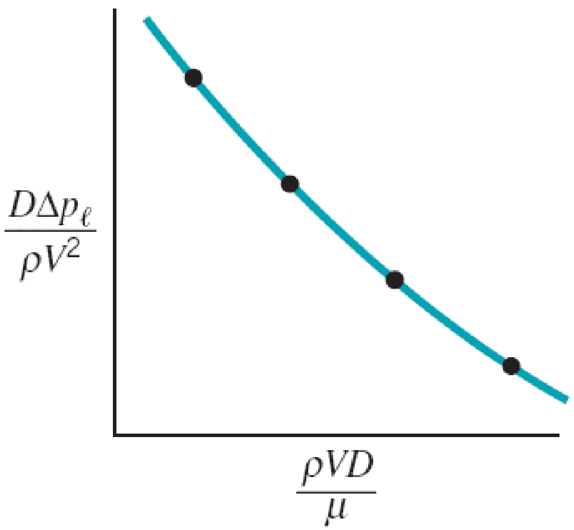
⇨

$$a = 1$$
$$b = -1$$
$$c = -1$$

⇩

$$\Pi_2 = \mu D^{-1} V^{-1} \rho^{-1} = \frac{\mu}{D\rho V}$$

$$\frac{\Delta p_l D}{\rho V^2} = \Phi\left(\frac{\mu}{D\rho V}\right)$$



Selection of variables

- Geometry: sufficient number of geometric variables should be included to describe the system e.g. length, angles, roughness scale etc.
- Material properties: viscosity, density
- External effects: variables that produce change in a system, e.g. pressure, velocity, gravity etc.

Uniqueness of Pi terms

- Set of Pi terms are **not** unique. However the required number of Pi terms is fixed and all other possible sets can be developed by combinations of products (with some powers) of Pi terms from a single set.

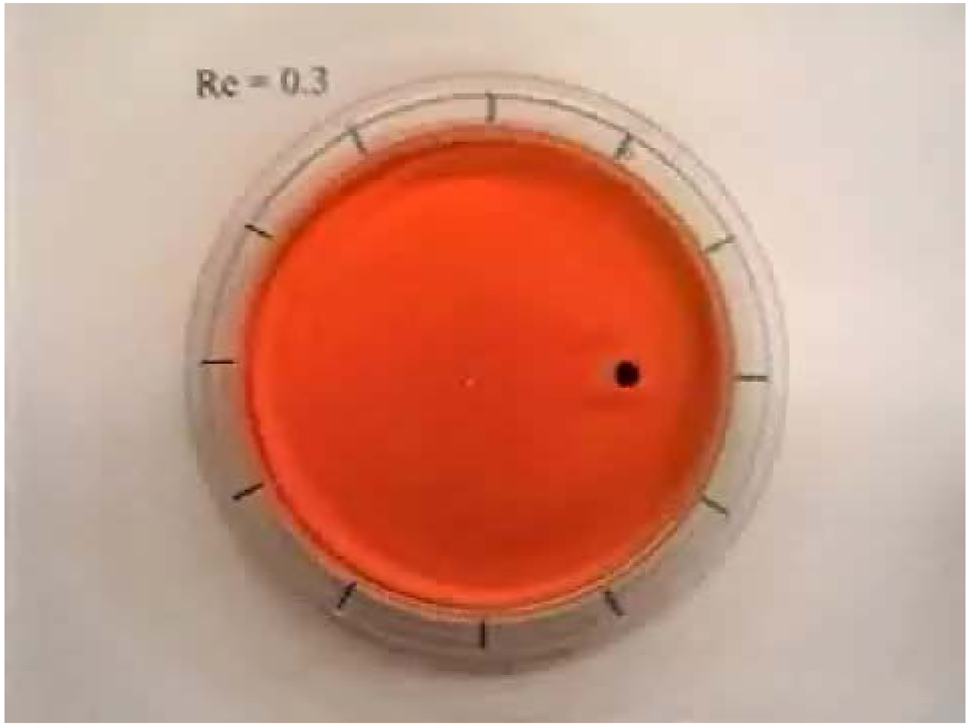
Common dimensionless groups in fluid mechanics

Variables: Acceleration of gravity, g; Bulk modulus, E_v; Characteristic length, ℓ; Density, ρ; Frequency of oscillating flow, ω; Pressure, p (or Δp); Speed of sound, c; Surface tension, σ; Velocity, V; Viscosity, μ

Dimensionless Groups	Name	Interpretation (Index of Force Ratio Indicated)	Types of Applications
$\dfrac{\rho V \ell}{\mu}$	Reynolds number, Re	$\dfrac{\text{inertia force}}{\text{viscous force}}$	Generally of importance in all types of fluid dynamics problems
$\dfrac{V}{\sqrt{g\ell}}$	Froude number, Fr	$\dfrac{\text{inertia force}}{\text{gravitational force}}$	Flow with a free surface
$\dfrac{p}{\rho V^2}$	Euler number, Eu	$\dfrac{\text{pressure force}}{\text{inertia force}}$	Problems in which pressure, or pressure differences, are of interest
$\dfrac{\rho V^2}{E_v}$	Cauchy number,[a] Ca	$\dfrac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\dfrac{V}{c}$	Mach number,[a] Ma	$\dfrac{\text{inertia force}}{\text{compressibility force}}$	Flows in which the compressibility of the fluid is important
$\dfrac{\omega \ell}{V}$	Strouhal number, St	$\dfrac{\text{inertia (local) force}}{\text{inertia (convective) force}}$	Unsteady flow with a characteristic frequency of oscillation
$\dfrac{\rho V^2 \ell}{\sigma}$	Weber number, We	$\dfrac{\text{inertia force}}{\text{surface tension force}}$	Problems in which surface tension is important

[a]The Cauchy number and the Mach number are related and either can be used as an index of the relative effects of inertia and compressibility. See accompanying discussion.

Reynolds number



- For a rotating tank containing a very viscous fluid, which gives a small Reynolds number, viscous forces are dominant. Thus, when the tank is suddenly stopped fluid particles also suddenly stop due to the dominance of viscous forces over inertia forces. Correspondingly, when a low viscosity fluid is in the tank, which gives a much higher Reynolds number, inertia forces are dominant. When the tank suddenly stops the fluid particles continue to move

Correlation of experimental data

- Dimensional analysis doesn’t provide the coefficients but can elucidate the important dependencies

Problems with One Pi term

$$\Pi_1 = C$$

- How drag D on a spherical particle that fall slowly through a viscous fluid will depend on particle diameter d, velocity V and fluid viscosity m?

$$D = f(d, V, \mu)$$

$$D \doteq F$$
$$d \doteq L$$
$$V \doteq LT^{-1}$$
$$\mu \doteq FL^{-2}T$$

$$\Pi_1 = \frac{D}{\mu V d} = C$$

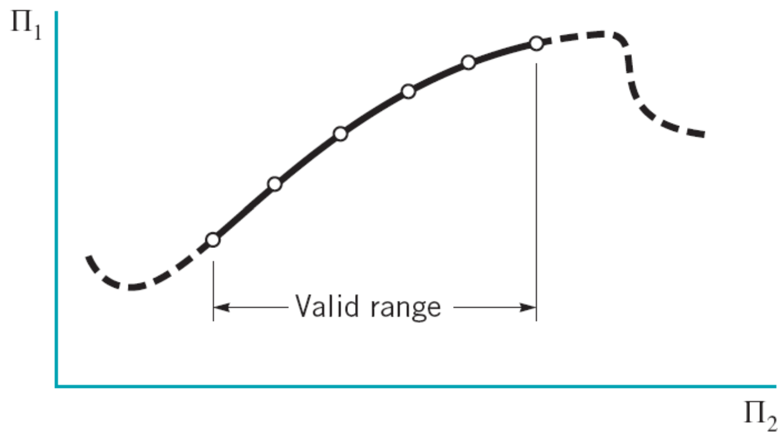
Stokes law (for spherical bodies):

$$D = 3\pi\mu V d$$

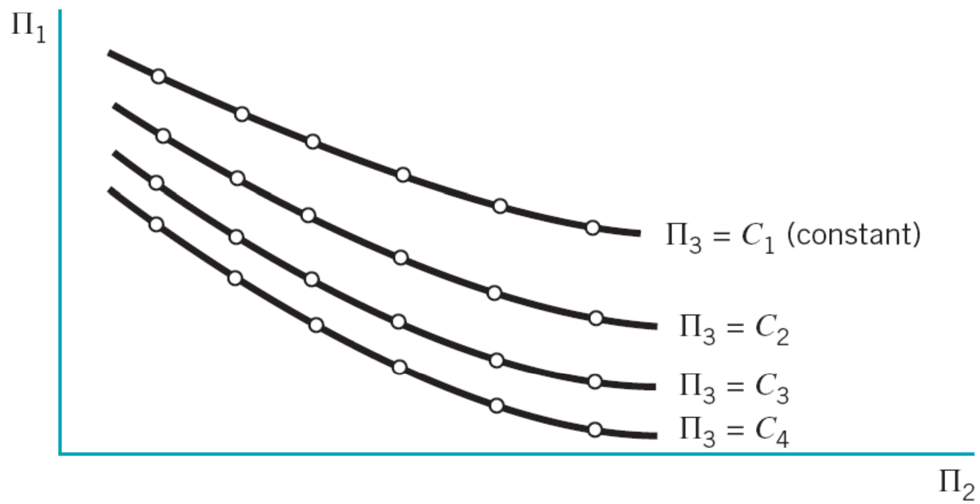
Valid for laminar flow as we didn’t include inertial effects (fluid density) in our variables

Problem with Two of more Pi terms

$$\Pi_1 = \Phi(\Pi_2)$$



For the problem involving 3 Pi terms one can still plot family of curves



Example: Blasius' equation

- Viscous flow was studied for a range of flow velocities

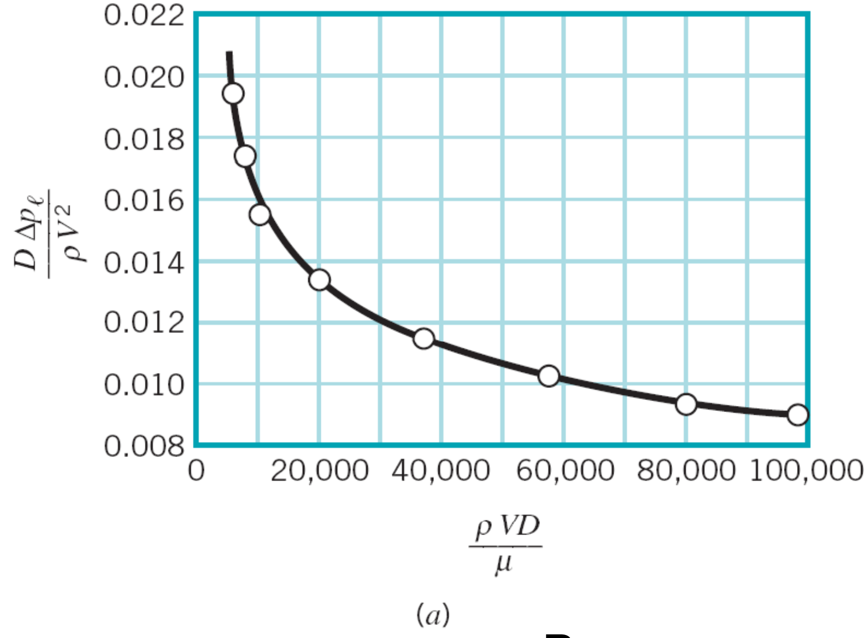
$$\Delta p_l = f(D, \rho, \mu, V)$$

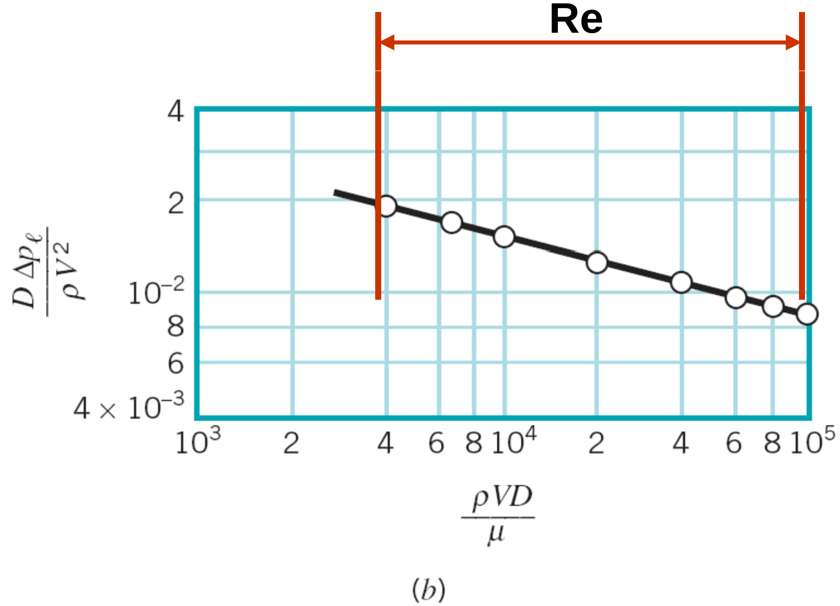
⇩

$$\frac{\Delta p_l D}{\rho V^2} = \Phi\left(\frac{\mu}{D\rho V}\right)$$

Blasius' equation:

$$\frac{\Delta p_l D}{\rho V^2} = 0.1582\left[\frac{D\rho V}{\mu}\right]^{-1/4}$$





Modeling and Simultitude

Suppose, a system can be described with a given set of Pi terms

$$\Pi_1 = \Phi(\Pi_2, \Pi_3, ..., \Pi_n)$$

if we can construct a model governed by the same variables:

$$\Pi_{1m} = \Phi(\Pi_{2m}, \Pi_{3m}, ..., \Pi_{nm})$$

and

$$\Pi_{2m} = \Pi_2$$

$$\Pi_{3m} = \Pi_3$$

......

$$\Pi_{nm} = \Pi_n$$

Than

$$\Pi_{1m} = \Pi_1$$

Prediction equation – indicates how to relate measured model data to the real system

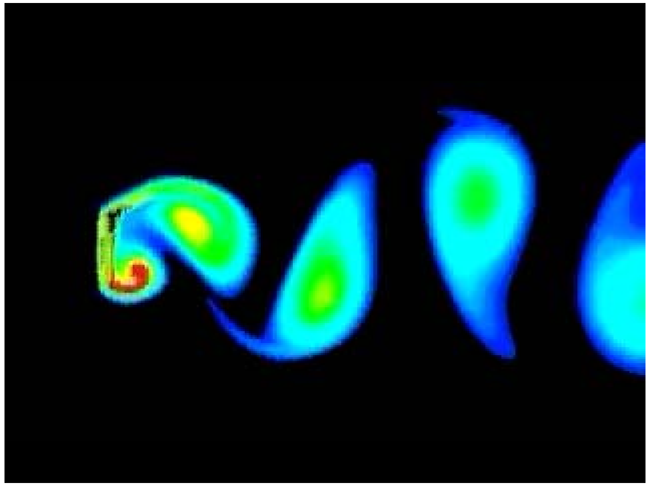
Similarity between a model and a prototype is achieved by equating the Pi terms

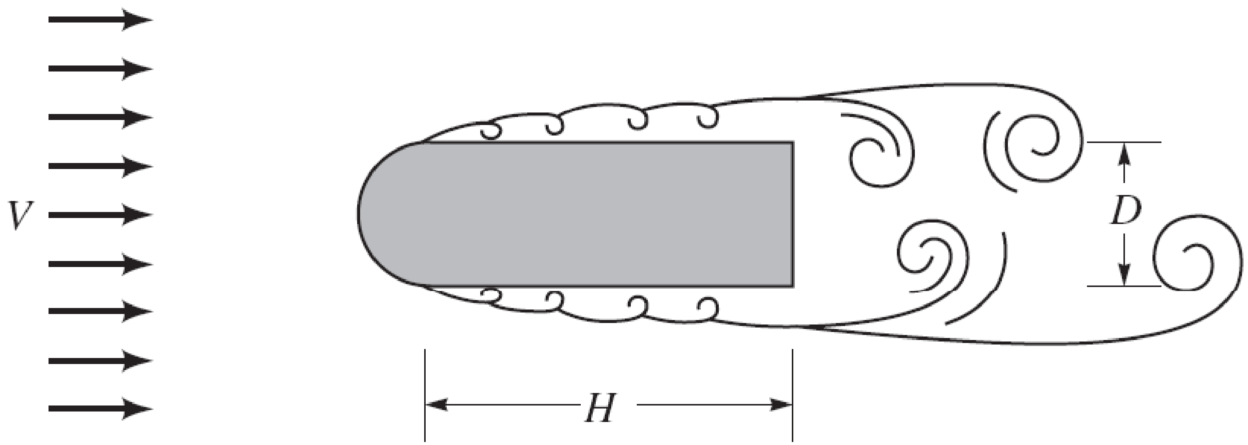
Environmental models

- Plume dispersion in a building complex is studied using scale models located in a large environmental wind tunnel. Spires at the tunnel entrance and roughness elements on the floor of the tunnel are used to create the necessary flow similarity in the test section. The effect of wind speed and direction on the dispersion of a plume can be determined for the geometrically scaled model. (Video courtesy of Cermak Peterka Petersen, Inc.)

Example: Prediction of prototype performance from Model Data (or Takoma Narrow Bridge effect)

Long structural component of bridge has cross section D=0.1m by H=0.3m and representative wind velocity is 50 km/h. Model with D=20mm was tested in water tunnel. What should be H_m? If shedding frequency for model was 49.9 Hz, what shell we expect from the prototype?





$$\omega = f(D, H, V, \rho, \mu)$$

$$\omega \doteq T$$
$$D \doteq L$$
$$H \doteq L$$
$$V \doteq LT^{-1}$$
$$\rho \doteq ML^{-3}$$
$$\mu \doteq ML^{-1}T^{-1}$$

$$\omega = f(D, H, V, \rho, \mu)$$

We have 6 variables, 3 reference dimensions, therefore 3 Pi terms are required

$$\frac{\omega D}{V} = \Phi\left(\frac{D}{H}, \frac{\rho V D}{\mu}\right)$$

To maintain the similarity:

$$\frac{D_m}{H_m} = \frac{D}{H}$$

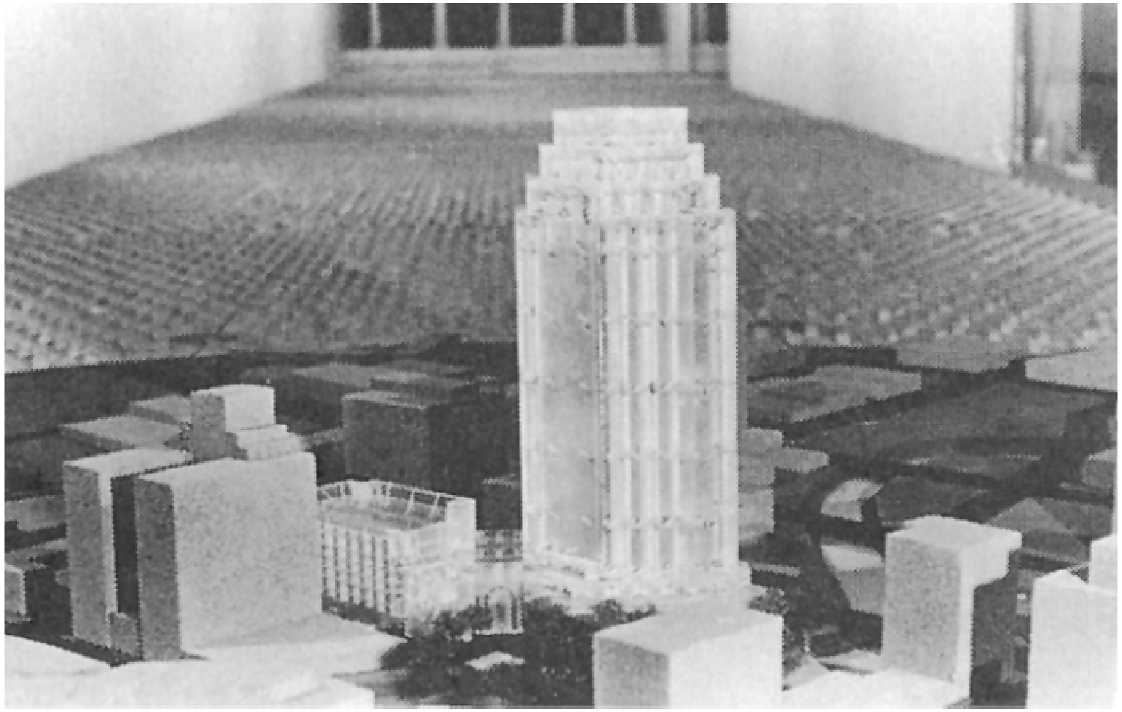
$$\frac{\rho_m V_m D_m}{\mu_m} = \frac{\rho V D}{\mu}$$

$$\frac{\omega D_m}{V_m} = \frac{\omega D}{V}$$

Practical Aspects of using Models

- Validation of model design: although the number of assumptions is smaller than in mathematical models, the assumptions might introduce uncertainty
- Scale: ratio of geometrical dimensions should be maintained
- *Distorted* Models: models for which one or more similarities requirements are not satisfied. Distorted models can be used but the interpretation of the results is more complicated that in case of *true* models

Wind Engineering models



Model of National Bank of commerce in San Antonio, Texas located in large meteorological wind tunnel

Simulitude based on Governing Differential equations

- If the governing differential equations are known the similarity can be established from there

$$\rho\frac{\partial\vec{v}}{\partial t}+\rho\left(\vec{v}\cdot\nabla\right)\vec{v}=-\nabla p+\rho\vec{g}+\mu\nabla^{2}\vec{v}$$

- Let's introduce dimensionless (normalized) variables

$$\vec{v}*=\vec{v}/v_0;\ \ \vec{r}*=\vec{r}/r_0;\ t*=t/\tau;$$

$$p*=p/p_0$$

$$\left[\frac{\rho V}{\tau}\right]\frac{\partial\vec{v}*}{\partial t}+\left[\frac{\rho V^2}{l}\right]\left(\vec{v}*\cdot\nabla\right)\vec{v}*=-\left[\frac{p_0}{l}\right]\nabla p+\rho\vec{g}+\left[\frac{\mu V}{l^2}\right]\nabla^2\vec{v}$$

Inertia force (local) | Inertia force (convective) | Pressure force | gravity force | Viscous force

$$\left[\frac{l}{\tau V}\right]\frac{\partial\vec{v}*}{\partial t}+\left(\vec{v}*\cdot\nabla\right)\vec{v}*=-\left[\frac{p_0}{\rho V^2}\right]\nabla p+\left[\frac{gl}{V^2}\right]\vec{g}*+\left[\frac{\mu_0}{\rho Vl}\right]\nabla^2\vec{v}$$

St | **Eu** | **$1/Fr^2$** | **1/Re**

Problems

- MYO 7.6
- MYO 7.16