Several applications of the Feshbach formula

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le 16 octobre 2008

Contents

Disclaimer: these notes do NOT contain completely rigorous proofs. They are intended for physics graduate students with no background in functional analysis but with a certain knowledge of advanced quantum mechanics as acquired through physics courses.

1 A formal proof of the Feshbach formula

Although the main reference is Feshbach's original article [F], the presentation in these notes is more general and considerably less mathematically inconsistent.

Let $\mathcal{H}$ be a Hilbert space (a vector space with an inner-product), and let H be a self-adjoint Hamilton operator. The example to bear in mind is

$$H = H_0 - V \tag{1}$$

where H_0 is a "known", solvable model, while $-V$ is a perturbation.

Let Π_{eff} be an orthogonal projector which commutes with H_0, and define $\Pi_\perp := 1 - \Pi_{\text{eff}}$. Then the Hilbert space admits a decomposition $\mathcal{H} = \mathcal{H}_{\text{eff}} \oplus \mathcal{H}_\perp$.

We are interested in writing H and its resolvent $(H-\xi)^{-1}$ as matrices of operators according to the above decomposition. With obvious notations:

$$H = \begin{pmatrix} H_{\text{eff}} & H_{\text{eff},\perp} \\ H_{\perp,\text{eff}} & H_{\perp,\perp} \end{pmatrix} = \begin{pmatrix} H_{\text{eff}} & -V_{\text{eff},\perp} \\ -V_{\perp,\text{eff}} & H_{\perp,\perp} \end{pmatrix},$$

where we used the fact that only V contributes to the off-diagonal parts.

The Feshbach formula. The resolvent is given by:

$$(H-\xi)^{-1} = \begin{pmatrix} S_W & S_W V R \\ R V S_W & R + R V S_W V R \end{pmatrix} \tag{2}$$

with

$$R(\xi) := \left[\Pi_\perp (H-\xi)\Pi_\perp\right]^{-1}, \quad W(\xi) = -\Pi_{\text{eff}} V R(\xi) V \Pi_{\text{eff}}, \quad S_W := (H_{\text{eff}} + W(\xi) - \xi)^{-1}. \tag{3}$$

We stress that $R(\xi)$ is the inverse of $\Pi_\perp(H-\xi)\Pi_\perp$ as an operator in $\mathcal{H}_\perp$.

Remark. Before proving the formula, let us explain what is the physical idea behind it. First, the resolvent contains all the physical information of the system: the numbers z where it becomes singular must lie in the spectrum. Second, if we know that the system is close to a "pure" state given by Π_{eff}, then the Feshbach formula provides us with a quantitative method of focusing on a certain spectral region. It is thus enough to solve a *non-linear*, effective problem, where the influence of the "rest of the world" modeled by $\Pi_\perp$ enters as a so-called self-energy $W(\xi)$.

Proof. Let

$$A = \begin{pmatrix} H_{\text{eff}} & 0 \\ 0 & H_{\perp,\perp} \end{pmatrix},$$

and

$$B = \begin{pmatrix} 0 & H_{\text{eff},\perp} \\ H_{\perp,\text{eff}} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -V_{\text{eff},\perp} \\ -V_{\perp,\text{eff}} & 0 \end{pmatrix},$$

where clearly $H = A + B$. Note that B is off-diagonal. Let us introduce some simplifying notation:

$$(A-\xi)^{-1} = \begin{pmatrix} (H_{\text{eff}} - \xi)^{-1} & 0 \\ 0 & (H_{\perp,\perp} - \xi)^{-1} \end{pmatrix} =: \begin{pmatrix} a & 0 \\ 0 & R \end{pmatrix}.$$

From the simple equality

$$H - \xi = A - \xi + B = [1 + B(A-\xi)^{-1}](A-\xi)$$

we can write (using the geometric series in the second identity):

$$(H-\xi)^{-1} = (A-\xi)^{-1}[1 + B(A-\xi)^{-1}]^{-1} = \sum_{n\geq 0} (-1)^n (A-\xi)^{-1} [B(A-\xi)^{-1}]^n. \tag{4}$$

The whole idea of the proof is to re-sum the above series in the form given in (2).

We start by summing separately with respect to even n and then odd n:

$$(H-\xi)^{-1}=\sum_{p\geq 0}(A-\xi)^{-1}[B(A-\xi)^{-1}]^{2p}-\sum_{p\geq 0}(A-\xi)^{-1}[B(A-\xi)^{-1}]^{2p}[B(A-\xi)^{-1}]. \quad (5)$$

Let us show that the first series contributes only to the diagonal terms in (2), while the second series generates the off-diagonal terms. The explanation is simple. We will show that $B(A-\xi)^{-1}$ is off-diagonal, but raised at an even power will become diagonal. At an odd power becomes again off-diagonal.

Indeed,

$$B(A-\xi)^{-1}=\begin{pmatrix}0 & H_{\text{eff},\perp}\\ H_{\perp,\text{eff}} & 0\end{pmatrix}\begin{pmatrix}a & 0\\ 0 & R\end{pmatrix}=\begin{pmatrix}0 & H_{\text{eff},\perp}R\\ H_{\perp,\text{eff}}a & 0\end{pmatrix}, \quad (6)$$

while

$$[B(A-\xi)^{-1}]^2=\begin{pmatrix}H_{\text{eff},\perp}RH_{\perp,\text{eff}}a & 0\\ 0 & H_{\perp,\text{eff}}aH_{\text{eff},\perp}R\end{pmatrix}. \quad (7)$$

Clearly, for any integer $p\geq 0$:

$$[B(A-\xi)^{-1}]^{2p}=\begin{pmatrix}[H_{\text{eff},\perp}RH_{\perp,\text{eff}}a]^p & 0\\ 0 & [H_{\perp,\text{eff}}aH_{\text{eff},\perp}R]^p\end{pmatrix}, \quad (8)$$

thus introducing it in the first series of (5) we get:

$$\sum_{p\geq 0}(A-\xi)^{-1}[B(A-\xi)^{-1}]^{2p}=\begin{pmatrix}\sum_{p\geq 0}a[H_{\text{eff},\perp}RH_{\perp,\text{eff}}a]^p & 0\\ 0 & \sum_{p\geq 0}R[H_{\perp,\text{eff}}aH_{\text{eff},\perp}R]^p\end{pmatrix}. \quad (9)$$

Now in analogy with (5) we see that:

$$\begin{aligned}\sum_{p\geq 0}a[H_{\text{eff},\perp}RH_{\perp,\text{eff}}a]^p &= [H_{\text{eff}}-\xi-H_{\text{eff},\perp}RH_{\perp,\text{eff}}]^{-1}\\ &=[H_{\text{eff}}-\xi+W]^{-1}=S_W. \quad (10)\end{aligned}$$

For the other diagonal element, we need one trick more. We write:

$$\begin{aligned}&\sum_{p\geq 0}R[H_{\perp,\text{eff}}aH_{\text{eff},\perp}R]^p=R+\sum_{p\geq 1}R[H_{\perp,\text{eff}}aH_{\text{eff},\perp}R]^p\\ &=R+\sum_{p\geq 1}R\underbrace{[H_{\perp,\text{eff}}aH_{\text{eff},\perp}R]\cdot[H_{\perp,\text{eff}}aH_{\text{eff},\perp}R]\dots[H_{\perp,\text{eff}}aH_{\text{eff},\perp}R]}_{p\text{ factors}}\\ &=R+RH_{\perp,\text{eff}}\sum_{p\geq 1}a\underbrace{[H_{\text{eff},\perp}RH_{\perp,\text{eff}}a]\dots[H_{\text{eff},\perp}RH_{\perp,\text{eff}}]a]}_{p-1\text{ factors}}H_{\text{eff},\perp}R\\ &=R+RH_{\perp,\text{eff}}\sum_{p\geq 0}a[H_{\text{eff},\perp}RH_{\perp,\text{eff}}a]^pH_{\text{eff},\perp}R\\ &=R+RH_{\perp,\text{eff}}S_WH_{\text{eff},\perp}R. \quad (11)\end{aligned}$$

Thus we have proved (see (8)):

$$\sum_{p\geq 0}(A-\xi)^{-1}[B(A-\xi)^{-1}]^{2p}=\begin{pmatrix}S_W & 0\\ 0 & R+RH_{\perp,\text{eff}}S_WH_{\text{eff},\perp}R\end{pmatrix}. \quad (12)$$

Now looking at the second series in (5) we realize that to compute it is enough to multiply the above series with the off-diagonal operator in (6). Thus:

$$-\sum_{p\geq 0}(A-\xi)^{-1}[B(A-\xi)^{-1}]^{2p+1} = \begin{pmatrix} 0 & -S_W H_{\text{eff},\perp} R \\ -RH_{\perp,\text{eff}}a - RH_{\perp,\text{eff}}S_W H_{\text{eff},\perp} RH_{\perp,\text{eff}}a & 0 \end{pmatrix}. \tag{13}$$

Now we are done with three terms in (2), and we only miss the one from bottom left. This last term would be obtained too, provided we could prove the following identity:

$$a + S_W H_{\text{eff},\perp} RH_{\perp,\text{eff}}a = S_W. \tag{14}$$

Exercise. Prove (14) using (10).

□

2 Perturbation theory: eigenvalues and resonances

We will now use the Feshbach formula in order to study how a possibly degenerate eigenvalue is perturbed by a "small" potential. The mathematical rigor will be minimal.

Let $\mathcal{H}$ be a Hilbert space, and H_0 a self-adjoint Hamilton operator. We assume that H_0 has discrete and continuous spectrum, and we can write:

$$H_0 = \sum_{k\geq 1} E_k|\psi_k\rangle\langle\psi_k| + \int_{\sigma_{\text{ac}}} E|\phi_E\rangle\langle\phi_E|dE. \tag{15}$$

Here ψ_k denotes a normalized eigenfunction belonging to $\mathcal{H}$, $\{E_k\}_{k\geq 1}$ are the possibly degenerate isolated eigenvalues, σ_{ac} is the (absolutely) continuous spectrum, while ϕ_E denotes a generalized eigenfunction.

The resolvent of H_0 can be expressed as:

$$(H_0 - z)^{-1} = \sum_{k\geq 1} \frac{1}{E_k - z}|\psi_k\rangle\langle\psi_k| + \int_{\sigma_{\text{ac}}} \frac{1}{E-z}|\phi_E\rangle\langle\phi_E|dE. \tag{16}$$

We can see that the resolvent is singular as a function of z when z hits the spectrum of H_0.

We are interested in what happens with a discrete eigenvalue of H_0 when we perturb H_0 with a potential λV, where λ is a small coupling constant.

Define $H_\lambda := H_0 + \lambda V$, and choose an eigenvalue Υ of H_0 having degeneracy n. This means that there exist n orthogonal eigenfunctions $\{\psi_k^{(0)}\}_{k=1}^n$ such that

$$H_0\psi_k^{(0)} = \Upsilon\psi_k^{(0)}, \quad k = 1, \dots, n.$$

Denote by

$$\Pi_{\text{eff}} := \sum_{k=1}^{n} |\psi_k^{(0)}\rangle\langle\psi_k^{(0)}|, \quad \Pi_{\text{perp}} := 1 - \Pi_{\text{eff}}. \tag{17}$$

We have two immediate identities:

$$\Pi_{\text{eff}} H_0 \Pi_{\text{eff}} = \Upsilon\Pi_{\text{eff}},$$

$$R_0(z) := [\Pi_{\text{perp}}(H_0 - z)\Pi_{\text{perp}}]^{-1} = \sum_{E_k\neq\Upsilon} \frac{1}{E_k - z}|\psi_k\rangle\langle\psi_k| + \int_{\sigma_{\text{ac}}} \frac{1}{E-z}|\phi_E\rangle\langle\phi_E|dE. \tag{18}$$

Now let us go back to (2) and (3), and identify the different quantities. Remember that we want to determine the singular ξ values for which the resolvent of H_λ does not exist. If V is a so-called regular perturbation, then the spectrum of H_λ near Υ will remain discrete, consisting of exactly n (possibly degenerate) eigenvalues $\nu_k(\lambda)$. According to the Feshbach formula, these eigenvalues are precisely those numbers ξ for which the n dimensional matrix

$$(\Upsilon - \xi)\Pi_{\text{eff}} + \lambda\Pi_{\text{eff}}V\Pi_{\text{eff}} - \lambda^2\Pi_{\text{eff}}V[\Pi_{\text{perp}}(H_0 - \xi + \lambda V)\Pi_{\text{perp}}]^{-1}V\Pi_{\text{eff}} \tag{19}$$

is NOT invertible.

2.1 The non-degenerate case

Assume that $n = 1$. Then $\Pi_{\text{eff}} = |\psi_1^{(0)}\rangle\langle\psi_1^{(0)}|$ is one dimensional, thus the unknown perturbed eigenvalue $\nu_1(\lambda)$ must be the unique solution near Υ of the nonlinear equation in ξ:

$$\xi = \Upsilon + \lambda\langle\psi_1^{(0)}, V\psi_1^{(0)}\rangle - \lambda^2\langle V\psi_1^{(0)}, [\Pi_{\text{perp}}(H_0 - \xi + \lambda V)\Pi_{\text{perp}}]^{-1}V\psi_1^{(0)}\rangle. \tag{20}$$

This equation can be solved in the following way. The general theory insures the fact that $\nu_1(\lambda)$ is analytic in λ near $\lambda_0 = 0$ thus can be written as an absolutely convergent power series $\sum_{m\geq 0} a_m\lambda^m$. if λ is small enough. We can determine the coefficients a_m by inserting the series in (20). For example, a_0 must be Υ. Then $a_1 = \langle\psi_1^{(0)}, V\psi_1^{(0)}\rangle$. Also, we get for free:

$$\begin{aligned} a_2 &= -\langle V\psi_1^{(0)}, R_0(\Upsilon)V\psi_1^{(0)}\rangle \\ &= -\sum_{E_k\neq\Upsilon} |\langle V\psi_1^{(0)}, \psi_k\rangle|^2(E_k - \Upsilon)^{-1} - \int_{\sigma_{\text{ac}}} dE|\langle V\psi_1^{(0)}, \psi_E\rangle|^2(E - \Upsilon)^{-1}. \end{aligned} \tag{21}$$

The third order correction becomes a bit more complicated. From (20), we see that it can only come from the term:

$$-\lambda^2\langle V\psi_1^{(0)}, [\Pi_{\text{perp}}(H_0 - \Upsilon - a_1\lambda + \lambda V)\Pi_{\text{perp}}]^{-1}V\psi_1^{(0)}\rangle$$

where we have to identify the coefficient of λ^3. For that, note the identity:

$$[\Pi_{\text{perp}}(H_0 - \Upsilon - a_1\lambda + \lambda V)\Pi_{\text{perp}}]^{-1} = R_0(\Upsilon) + \lambda a_1 R_0(\Upsilon)^2 - \lambda R_0(\Upsilon)VR_0(\Upsilon) + \mathcal{O}(\lambda^2). \tag{22}$$

Thus:

$$a_3 = -a_1\langle V\psi_1^{(0)}, R_0(\Upsilon)^2V\psi_1^{(0)}\rangle + \langle V\psi_1^{(0)}, R_0(\Upsilon)VR_0(\Upsilon)V\psi_1^{(0)}\rangle. \tag{23}$$

Therefore we see that we can determine all coefficients in a recursive way. This method is in fact the most efficient in computing the Rayleigh-Schrödinger series of a perturbed eigenvalue. It also provides an elegant proof of the so-called linked diagram theorem of Goldstone [G].

Exercise. Compute a_4.

2.2 The case of an embedded eigenvalue

Now assume that Υ is an eigenvalue immersed in the continuous spectrum of H_0. The main question is the following: does the eigenvalue survive after turning on the perturbation λV, or does it disappear?

The generic fact in this case is that the eigenvalue disappears. Let us argue why, although the argument below lacks any reference to the Limiting Absorption Principle.

As before, if there exists a eigenvalue for H_λ near Υ, it must solve equation (20) and must be **real**. We do the following trick: for complex numbers of the form $x + i\epsilon$ with x near Υ and $\epsilon > 0$ we define the map:

$$F(x + i\epsilon) = \Upsilon + \lambda\langle\psi_1^{(0)}, V\psi_1^{(0)}\rangle - \lambda^2\langle V\psi_1^{(0)}, [\Pi_{\text{perp}}(H_0 - x - i\epsilon + \lambda V)\Pi_{\text{perp}}]^{-1}V\psi_1^{(0)}\rangle. \tag{24}$$

Now if we expand this in powers of λ up to the second order we obtain:

$$F(x + i\epsilon) = \Upsilon + \lambda\langle\psi_1^{(0)}, V\psi_1^{(0)}\rangle - \lambda^2\langle V\psi_1^{(0)}, [\Pi_{\text{perp}}(H_0 - x - i\epsilon)\Pi_{\text{perp}}]^{-1}V\psi_1^{(0)}\rangle + \mathcal{O}(\lambda^3), \tag{25}$$

where one can show that for "nice" V's the remainder is uniform in $\epsilon > 0$. By "nice" we mean for example rapid decay.

Now let us compute the imaginary part of $F(x + i\epsilon)$ in the limit $\epsilon \searrow 0$. If it is non-zero, then no real x can solve (20), thus the eigenvalue must disappear.

We compute:

$$\begin{aligned}\mathrm{Im}(F(x+i\epsilon)) &= -\lambda^2 \mathrm{Im} \int_{\sigma_{\mathrm{ac}}} dE |\langle V\psi_1^{(0)}, \psi_E\rangle|^2 (E-x-i\epsilon)^{-1} + \mathcal{O}(\lambda^3) \\ &= -\lambda^2 \int_{\sigma_{\mathrm{ac}}} dE |\langle V\psi_1^{(0)}, \psi_E\rangle|^2 \frac{\epsilon}{(E-x)^2+\epsilon^2} + \mathcal{O}(\lambda^3). \end{aligned} \tag{26}$$

By taking ϵ to zero, we obtain the identity:

$$\lim_{\epsilon \searrow 0} \mathrm{Im}(F(x+i\epsilon)) = -\pi\lambda^2 |\langle V\psi_1^{(0)}, \psi_x\rangle|^2 + \mathcal{O}(\lambda^3). \tag{27}$$

Thus if $\langle V\psi_1^{(0)}, \psi_x\rangle \neq 0$, i.e. if V couples the eigenfunction $\psi_1^{(0)}$ with the generalized eigenfunctions corresponding to the continuous spectrum near Υ, then no real x can solve (20). In fact, it is enough to have $\langle V\psi_1^{(0)}, \psi_\Upsilon\rangle \neq 0$ in order to be sure that (20) has no real solution.

One can solve (20) by searching **complex** solutions. It is sometimes possible to show that there exists a unique such complex solution, called resonance, which has a negative imaginary part.

3 Excitons on the surface of a cylinder

We start with the model that one can find in [CDR, KCM, P]. Let $\mathcal{C} := \mathbb{R} \times rS^1$ denote the cylinder of radius r; we mean the surface. The Hilbert space $L^2(\mathcal{C})$ will be represented in the trivial chart by $\mathcal{H} := L^2(\Omega)$ with $\Omega := \mathbb{R} \times (-\pi r, \pi r)$, with periodic boundary conditions, $\psi(x, \pi r) = \psi(x, -\pi r)$, (all $x \in \mathbb{R}$), whenever ψ is continuous. We consider the Hamiltonian

$$H := -\frac{\Delta}{2} - V, \quad H_0 = -\frac{\Delta}{2}, \quad V(x,y) = V^r(x,y) := \frac{1}{\sqrt{x^2 + 4r^2 \sin^2(\frac{y}{2r})}} \tag{28}$$

acting in $L^2(\Omega)$; H denotes the self-adjoint realization characterized by its quadratic form domain[1]

$$Q(H) := \left\{\psi, \in \mathcal{H}^1(\Omega), \psi(\cdot, \pi r) = \psi(\cdot, -\pi r)\right\}.$$

Next we consider that $H_0 = H_{0,l} \otimes 1 + 1 \otimes H_{0,t}$, with $H_{0,l} = -\frac{1}{2}\partial_x^2$ with domain $\mathcal{H}^2(\mathbb{R})$ and $H_{0,t} = -\frac{1}{2}\partial_y^2$ with domain $\{u \in, \mathcal{H}^2((-\pi r, \pi r)), u(\pi r) = u(-\pi r), u'(\pi r) = u'(-\pi r)\}$. The spectral decomposition of $H_{0,t}$ is:

$$H_{0,t} = \bigoplus_{n \in \mathbb{Z}_+} \frac{n^2}{2r^2} \Pi_n^r \tag{29}$$

where $\mathbb{Z}_+ := \{0, 1, 2, \ldots\}$ and Π_n^r denotes the eigenprojector on the span of

$$\chi_n^r(y) := \frac{1}{\sqrt{2\pi r}} e^{\pm i n y \frac{1}{r}}.$$

Notice that Π_0^r is one dimensional. We then have the following decomposition of $\mathcal{H}$:

$$\mathcal{H} := \mathcal{H}_{\mathrm{eff}} \oplus \mathcal{H}_\perp, \quad \text{with} \quad \mathcal{H}_{\mathrm{eff}} := \mathrm{Ran}\, 1 \otimes \Pi_0^r$$

and we want to consider the effective Hamiltonian

$$H_{\mathrm{eff}}^r := \amalg_{\mathrm{eff}} H \amalg_{\mathrm{eff}}, \quad \amalg_{\mathrm{eff}} := 1 \otimes \Pi_0^r.$$

There are several questions one could address:

Questions.

[1] $\mathcal{H}^s(\Omega)$ denotes the usual Sobolev space on the open set Ω. $Q(T)$ will always denote the quadratic form domain of T, for T s.a.

1. Does one know that H^r has always, i.e. for every $r > 0$, one negative bound state?
2. What is the deviation of H^r_{eff} from H?
3. What is the "limit" of H^r_{eff} as $r \to 0$? These are the natural models we are looking for!
4. Comparison of the natural models with the truncated coulomb model: $(|x| + a(r))^{-1}$. How can one choose the best $a(r)$?
5. What are the informations we can extract on the spectral properties of H from the ones of these models?
6. Study the resonances of H coming from the other effective Hamiltonians obtained by projecting on higher transverse modes.

3.1 H^r_{eff} and its "limit"

3.1.1 Definition of H^r_{eff}

Since Π^r_0 is the projection on constant functions (with respect to the y variable), we know at once that $\amalg_{\text{eff}} H_0 \amalg_{\text{eff}}$ is unitarily equivalent to $H_{0,l}$, and $\amalg_{\text{eff}} V \amalg_{\text{eff}}$ is unitarily equivalent, with the same unitary operator to V^r_{eff} acting on $L^2(\mathbb{R})$ with

$$
\begin{aligned}
V^r_{\text{eff}}(x) &= \int_{-\pi r}^{\pi r} V(x,y)\chi^r_0(y)^2 dy = \frac{1}{2\pi r}\int_{-\pi r}^{\pi r} \frac{dy}{\sqrt{x^2 + 4r^2\sin^2(\frac{y}{2r})}} \\
&= \frac{1}{2\pi r}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dy}{\sqrt{\left(\frac{x}{2r}\right)^2 + \sin^2(y)}} \\
&= \frac{1}{2r} V^{\frac{1}{2}}_{\text{eff}}\left(\frac{x}{2r}\right).
\end{aligned} \tag{30}
$$

It follows with $r = \alpha$ that

$$2\alpha V^{\alpha}_{\text{eff}}(2\alpha x) = V^{\frac{1}{2}}_{\text{eff}}(x)$$

and therefore:

$$\forall \alpha > 0, \qquad V^r_{\text{eff}}(x) = \frac{\alpha}{r} V^{\alpha}_{\text{eff}}\left(\frac{\alpha x}{r}\right)$$

This last equality is useful for what we are aiming for: this is a scaling property reminiscent from the homogeneous character of the Coulomb potential. We shall consider the particular case $\alpha = 1$ in the sequel:

$$\forall r > 0, \forall x \in \mathbb{R}, \qquad V^r_{\text{eff}}(x) = \frac{1}{r} V^1_{\text{eff}}\left(\frac{x}{r}\right). \tag{31}$$

It turns out that V^1_{eff} may be computed in terms of special functions:

$$V^1_{\text{eff}}(x) = \frac{2K(-\frac{4}{x^2})}{\pi|x|}$$

where K denotes the complete elliptic integral of the first kind [2]. It has the following asymptotic expansion at $+\infty$ (done with Mathematica):

$$V^1_{\text{eff}}(x) = \frac{1}{x} - \frac{1}{x^3} + \mathcal{O}(x^{-4}).$$

Exercise. Show that the behavior of V_{eff} at 0 is logarithmic. We shall see that:

$$V^1_{\text{eff}}(x) \overset{x\to 0}{=} -\frac{\log|x|}{\pi} + \frac{\log 2\pi}{\pi} + \mathcal{O}(x^2) \tag{32}$$

[2] $K(m) = \int_0^1((1-t^2)(1-mt^2))^{-\frac{1}{2}} dt$

We conclude from the properties which are announced in the above exercise that $H^r_{\rm eff}$ is unitarily equivalent to $-\frac{1}{2}\Delta - V_{\rm eff}$ acting in $L^2(\mathbb{R})$ with operator domain $\mathcal{H}^2(\mathbb{R})$. We also denote by $H^r_{\rm eff}$ this operator:

$$H^r_{\rm eff} = -\frac{1}{2}\Delta - V^r_{\rm eff}, \quad \operatorname{dom} H^r_{\rm eff} = \mathcal{H}^2(\mathbb{R}). \tag{33}$$

3.1.2 The delta model

Let $H_0 := -\frac{1}{2}\Delta$ and $H_\delta := H_0 - g\delta$. Let

$$\tau : \mathcal{H}^1 \to \mathcal{H}^{-1}, \quad \tau\psi := \psi(0).$$

Then

$$H_\delta = H_0 - g\tau^\star\tau.$$

So with the resolvent equation we get

$$R_\delta = R_0 + gR_0\tau^\star\tau R_\delta$$

so that

$$R_\delta = R_0 + \frac{g}{1 - g\tau R_0\tau^\star} R_0\tau^\star\tau R_0$$

which is the Krein's formula. To recover the ground state wave function φ_δ we use the equation

$$1 - g\tau R_0(E_\delta)\tau^\star = 0 \tag{34}$$

and proceed as follows

$$\begin{aligned} 1 - g\tau R_0(\zeta)\tau^\star &= 1 - g\tau R_0(\zeta)\tau^\star - (1 - g\tau R_0(E_\delta)\tau^\star) \\ &= g\tau(R_0(E_\delta) - R_0(\zeta))\tau^\star \\ &= g(E_\delta - \zeta)\tau R_0(E_\delta)R_0(\zeta)\tau^\star \end{aligned}$$

so that

$$R_\delta(\zeta) = R_0(\zeta) + \frac{1}{E_\delta - \zeta}\frac{R_0(\zeta)\tau^\star\tau R_0(\zeta)}{\tau R_0(E_\delta)R_0(\zeta)\tau^\star}$$

which shows that the eigenprojector on φ_δ is

$$P_\delta = \frac{R_0(E_\delta)\tau^\star\tau R_0(E_\delta)}{\tau R_0(E_\delta)^2\tau^\star}.$$

and therefore φ_δ may be identified with

$$\varphi_\delta \sim \frac{R_0(E_\delta)\tau^\star}{\sqrt{\tau R_0(E_\delta)^2\tau^\star}} \quad \text{or} \quad \frac{\tau R_0(E_\delta)}{\sqrt{\tau R_0(E_\delta)^2\tau^\star}}.$$

One has finally:

$$\varphi_\delta(x) = \frac{G_0(x, 0; E_\delta)}{\sqrt{\partial_\zeta G_0(0, 0; E_\delta)}} \tag{35}$$

since $R_0(\zeta)^2 = \partial_\zeta R_0(\zeta)$ where $G_0(x, x'; \zeta)$ denotes the Green's function of H_0:

$$G_0(x, x'; \zeta) := \frac{e^{-\sqrt{-2\zeta}|x-x'|}}{\sqrt{-2\zeta}}.$$

Since $E_\delta = -\frac{1}{2}g^2$ we get that $G_0(x, 0; E_\delta) = \frac{e^{-g|x|}}{g}$. Then

$$\partial_\zeta G_0(0, 0; E_\delta) = g^{-3}$$

3.1.3 "Limit" of $H^r_{\rm eff}$: the leading term

We apply the general theorem of [BD1, BD3]. Assume that V has a Fourier transform in the sense of tempered distributions $\hat{V}$ such that

- (1) $\hat{V} \in C^0(\mathbb{R} \setminus \{0\})$
- (2) there exists a constant c_0 such that
$$\sqrt{2\pi}\hat{V}(p) \overset{p\to 0}{=} -c_0 \log(|p|) + \mathcal{O}(1)$$
- (3) $\hat{V} \in L^\infty(\{|p| \geq \frac{1}{2}\})$.

then if $V_\lambda(x) := \lambda V(\lambda x)$

$$\forall \lambda_0, \alpha_0 > 0, \exists C > 0, \forall \lambda \geq \lambda_0, \forall \alpha > \alpha_0, \quad \|V_\lambda - \log \lambda c_0 \delta\|^2_{-1,1} \leq C\left(\frac{\log^2(\alpha)}{\alpha^2} + \frac{1}{\alpha\lambda}\right) \tag{36}$$

where $\|\cdot\|_{-1,1}$ denotes the α dependent norm of operators $X : \mathcal{H}^1 \to \mathcal{H}^{-1}$ defined by

$$\|X\|_{-1,1} := \|(D^2 + \alpha^2)^{-\frac{1}{2}} X (D^2 + \alpha^2)^{-\frac{1}{2}}\|. \tag{37}$$

Define
$$H^r_\delta := -\frac{1}{2}\Delta + \log r^2\, \delta \quad \text{with} \quad Q(H^r_\delta) := \mathcal{H}^1(\mathbb{R});$$
we recall that if $r < 1$ then H^r_δ has a unique eigenvalue $E_\delta(r) := -\frac{1}{2}(\log r^2)^2$ with the associated eigenvector
$$\varphi^r_\delta(x) := (2|E_\delta(r)|)^{\frac{1}{4}}\, e^{-\sqrt{2|E_\delta(r)|}\,|x|}. \tag{38}$$
Thus we may state now the

theorem δ model. *Let $\alpha(r) := \sqrt{2}|\log r^2|$ and $d_\delta(\zeta) := \mathrm{dist}\,(\zeta, \mathrm{spect}\, H_\delta)$. There exists $r_\delta > 0$ such that if $d_\delta(\zeta) \geq c_\delta \alpha^2$ and $0 < r < r_\delta < 1$ one has $\zeta \in \rho(H^r_{\rm eff})$ and*

$$\|(H^r_{\rm eff} - \zeta)^{-1} - (H_\delta - \zeta)^{-1}\| \leq \frac{C_\delta}{c_\delta}\frac{\log\alpha}{\alpha}\frac{1}{d_\delta(\zeta)}.$$

Here C_δ is a constant which depends only on $V^1_{\rm eff}$.

As a consequence one gets that: for all $0 < r < r_\delta < 1$, $E_{\rm eff}(r) := \inf H^r_{\rm eff}$ is an isolated eigenvalue of $H_{\rm eff}$ and moreover

$$E_{\rm eff}(r) - E_\delta(r) = \mathcal{O}\left(\log\frac{1}{r}\log_2\frac{1}{r}\right) \quad \text{and} \quad \|\varphi^r_{\rm eff} - \varphi^r_\delta\| = \mathcal{O}\left(\frac{\log_2 \frac{1}{r}}{\log\frac{1}{r}}\right)$$

if $\varphi^r_{\rm eff}$ denotes a properly chosen associated eigenvector of $H^r_{\rm eff}$.

3.2 Feshbach reduction of H to $H^r_{\rm eff}$

Let $\Pi_\perp$ denote the orthogonal projection on $\mathcal{H}_\perp := \mathcal{H}^\perp_{\rm eff}$:

$$\Pi_\perp := 1 - \Pi_{\rm eff}.$$

Theorem Feshbach 1. *There exists r_{39} such that if $\xi \leq 0$ and $r \leq r_{39}$ then ξ belongs the the resolvent set of $\Pi_\perp H \Pi_\perp$. If in addition $\xi \notin \mathrm{spect}\, H_{\rm eff} + W$ then ξ is in the resolvent set of H and*

$$\|(H - \xi)^{-1} - S_W \oplus 0.\Pi_\perp\| \leq \left(\frac{C_{40}}{d_W(\xi)} + 4r + \frac{1}{d_W(\xi)}C^2_{40} r\right) r.$$

We have used the following notations: $d_W(\xi) := \mathrm{dist}\,(\xi, \mathrm{spect}\,(H_{\mathrm{eff}} + W))$,

$$r_{39} := \frac{1}{2}\|\mathrm{I\!I}_\perp H_0^{-\frac{1}{2}} V H_0^{-\frac{1}{2}} \mathrm{I\!I}_\perp\|^{-1} \tag{39}$$

$$C_{40} := 2\sqrt{2}\|\mathrm{I\!I}_{\mathrm{eff}} V H_0^{-\frac{1}{2}} \mathrm{I\!I}_\perp\| \tag{40}$$

where the two last quantities are evaluated at $r = 1$.

Reduction to H_{eff}. *There exists* r_0*,* c_0 *and* C_0 *which depend only on* V_{eff} *such that if* $r < r_0$ *and* $\xi < 0$ *satisfy*

$$\beta^2(r) \geq d_0(\xi) \geq c_0 r \beta(r)$$

with $\beta(r) := \mathrm{pl}(r^{-1})$ *then* $\xi \in \rho(H)$*, and*

$$\|(H - \xi)^{-1} - (H_{\mathrm{eff}} - \xi)^{-1} \oplus 0 \cdot \mathrm{I\!I}_\perp\| \leq C_0 \frac{r\beta(r)^2}{d_0(\xi)^2}.$$

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